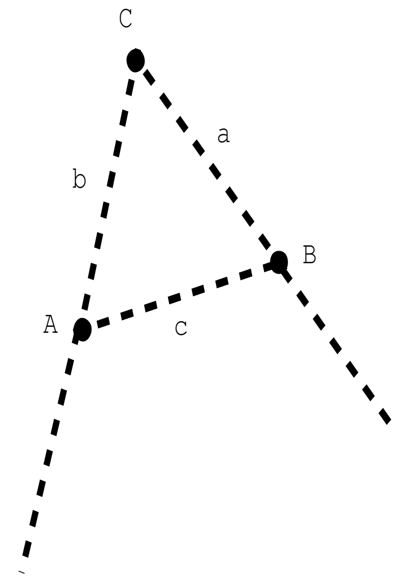

Figure 1: Part of a Triangulation

1 Closest Obstacle Calculation

In order to determine the size of the largest unit with a valid path between (unconstrained) edges a and b of a triangle in a Constrained (Delaunay) Triangulation, one can equivalently find the closest obstacle (vertex or point on a constrained edge) to the vertex these edges share (vertex C in this case) within the region extending from vertex C between edges a and b. This configuration is shown in Figure 1.

There are three cases possible within a triangle which can determine the closest obstacle in this region. The first such case is that either angle A or angle B is a right angle or obtuse (Section 1.1), the second arises when these angles are accute and edge c is constrained (Section 1.2), and finally the last possibility is when angles A and B are accute and edge c is unconstrained (Section 1.3).

Pseudocode for the algorithm which determines the closest obstacle between edge a and b in triangle T is given in code listings 1, 2, and 3. The proofs that this technique is equivalent to finding the maximum radius of a unit with a valid path through this triangle are given in Section 2.

In these proofs, we assume circular units, but this can apply to other shapes as well. The maximum allowable size through a series of adjacent triangles yields a winding path though them and can be used to determine the throughput of smaller units, or the maximum allowable size of a rectangular unit, for example.

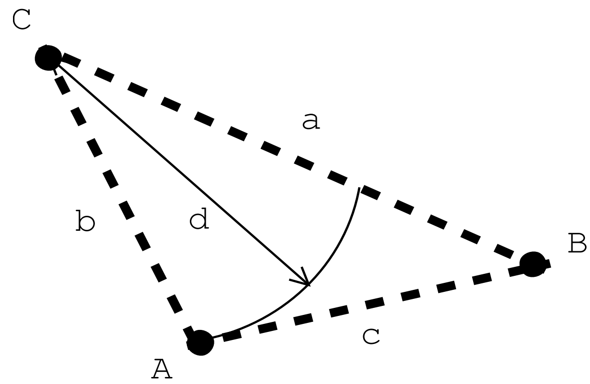

Figure 2: Case 1: Angle at Vertex A is Obtuse

The techniques used assume that at the very least, each vertex in a triangulation represents a contraint. That is, if one was representing an environment, one would only add a vertex to the triangulation either because it was an endpoint for a constrained edge, or if it were a point obstacle.

This is intuitive because adding unneeded vertices would only complicate the triangulation and slow down subsequent algorithms. If such vertices were added, however, these methods might incorrectly determine the maximum diameter of a unit through a path in the case where a path for the true largest possible unit would pass through such an unconstrained vertex.

In each case, the path for the unit of maximum diameter is determined as an arc hugging vertex C. While the unit need not always follow this path to traverse the triangle, it is true that a unit couldn't successfully traverse some other path and not this one, as is proven below.

1.1 Case 1: Angle A or B is Right or Obtuse

When one of angles A or B is right or obtuse, the first case applies. Assume, without loss of generality, that the angle at vertex A is right or obtuse. It follows that edge b is shorter than edge a. Thus the maximum allowable diameter d of a circular unit between edges a and b in this triangle is the length of edge a. See Figure 2 for a visual explanation.

This follows from the fact that the closest point on a line to a point is where a line passing through the point intersects the line at a right angle. The proof of this follows from the fact that the length of any other line intersecting both the point and the line is necessarily longer than this.

In Figure 3, the point c is the point where a line passing through point p intersects with line ab at a right angle. For any other point c' on the line ab, it will be some positive distance l away from c. Thus if the distance from p to c is d, then the distance (d') from p to c' is $\sqrt{d^2 + l^2} > d$. Thus c is the

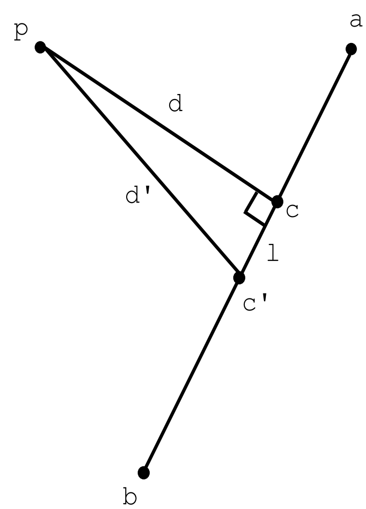

Figure 3: The Closest Distance of a Line to a Point

closest point to p on line ab, as desired.

Similarly, consider Figure 4. The length of segment b is $\sqrt{|CP|^2 + |PA|^2}$. For any point $A' \neq A$ along the segment between vertex A and vertex B, the length of the segment between C and A' would be $\sqrt{|CP|^2 + (|PA| + |AA'|)^2} > \sqrt{|CP|^2 + |PA|^2}$ since $|AA'| > 0$. Thus vertex A is the closest point on segment AB to vertex C, and since there can be no obstacles in this triangle (any obstacles would have been incorporated in the triangulation), it follows that the closest obstacle is $|b|$ away from vertex C and this is the maximum diameter of a unit that can traverse from edge a to edge b in this triangle.

1.2 Case 2: Edge c is Constrained

In the case that the interior angles at both vertices A and B are accute, the point on the line passing through the vertices A and B that is closest to vertex C lies between A and B as shown in Section 1.1 above. In the case that edge c is constrained, this point is an obstacle. This situation is shown in Figure 5 above.

As described in Section 1.1, since there can be no obstacles within the triangle, the closest point on edge c to vertex C (when edge c is constrained) represents the closest obstacle to vertex C in the triangle. Assuming the distance between vertex C and the point P on segment AB which makes CP perpendicular to AB, is d, the diameter of the largest circular unit that can traverse the triangle from edge a to edge b is d, as desired.

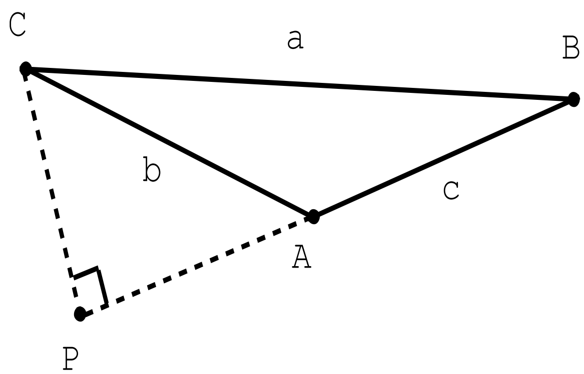

Figure 4: Triangle with One Obtuse Angle

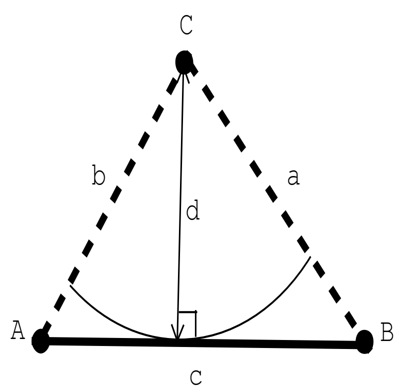

Figure 5: Case 2: Angles at Vertices A and B are Accute and Edge c is Constrained

1.3 Case 3: Edge c is Unconstrained

In the case where edge c is not constrained and the angles at both vertices A and B are accute, the situation gets slightly more complex, as the closest point on segment AB to vertex C no longer represents an obstacle.

Vertices A and B are still obstacles, and thus the maximum unit diameter that can traverse this triangle from edge a to edge b is bounded above by both $|a|$ and $|b|$. However, since there may still be obstacles on the opposite side of edge c from vertex C closer to C than either A or B, we must consider these possibilities.

What must occur then is a search that is bounded by the closest obstacle found so far. Thus this search begins by searching across edge c to the triangle opposite this edge, bounded above by $min\{|a|, |b|\}$ and continues as described below.

When the search enters a triangle via an edge, it checks the other two edges as follows. We will say each edge is the segment between two vertices U and V. First of all, an edge will only be considered if both angles $\angle CVU$ and $\angle CUV$ are accute, that is, if the closest point on the line passing through U and V to vertex C lies between these two points. If this criterion is not met, search along this branch ends.

Next, we consider the distance from vertex C to the closest point on edge UV. If this distance is greater than the current upper bound, search along this branch returns because further search will not yield a closer obstacle than the closest already found. If this distance is less than the current upper bound and edge UV is constrained, then the current upper bound is updated to reflect this new distance, and search returns from this branch. Finally if the distance is less than the current upper bound and UV is unconstrained, search continues across this edge.

2 Maximum Unit Diameters

In this section, we show that the distance to the closest obstacle between edges a and b is equivalent to the diameter of the largest circular unit with a valid path between these edges. In Section 2.1, we define what constitutes a valid path and go through some preliminary proofs that we will use in the sections that follow.

In Section 2.2, we prove that if the closest obstacle between edges a and b is at distance d from vertex C then there exists a valid path between these edges for a circular unit of diameter d, thus the method is sound.

Then in Section 2.3, we prove that if there exists any valid path for a unit of a given diameter, there will be no obstacles within this distance of vertex C between edges a and b, and thus the method is complete. These combined prove the equivalence of the distance between vertex C and the closest obstacle between edges a and b and the diameter of the largest unit

Algorithm 1 DistanceBetween(Vertex C, Edge e)

A , $B \leftarrow$ EndpointsOf(e)
if $A_x = B_x$ **then**
  $a \leftarrow 1$
  $b \leftarrow 0$
  $c \leftarrow -(C_x)$
else
  $rise \leftarrow B_y - A_y$
  $run \leftarrow B_x - A_x$
  $intercept \leftarrow A_y - (\frac{rise}{run})A_x$
  $a \leftarrow rise$
  $b \leftarrow -run$
  $c \leftarrow run \times intercept$
end if
return $\frac{|a \cdot C_x + b \cdot C_y + c|}{\sqrt{a^2+b^2}}$

Algorithm 2 SearchWidth(Vertex C, Triangle T, Edge e, Distance d)

$U, V \leftarrow$ EndpointsOf(e)
if IsObtuse(C, U, V) $\vee$ IsObtuse(C, V, U) **then**
  return d
end if
$d' \leftarrow$ DistanceBetween(C, e)
if $d' > d$ **then**
  return d
else if IsConstrained(e) **then**
  return d'
else
  $T' \leftarrow$ TriangleOpposite(T, e)
  $e', e'' \leftarrow$ OtherEdges(T', e)
  $d \leftarrow$ SearchWidth(C, T', e', d)
  return SearchWidth(C, T', e'', d)
end if

Algorithm 3 CalculateWidth(Triangle T, Edge a, edge b)

$C \leftarrow$ VertexBetween(a, b)
$c \leftarrow$ EdgeOpposite(C, T)
$A \leftarrow$ VertexOpposite(a, T)
$B \leftarrow$ VertexOpposite(b, T)
$d \leftarrow \min\{$Length(a), Length(b)$\}$
if IsObtuse(C, A, B) $\vee$ IsObtuse(C, B, A) **then**
  return d {Case: 1}
else if IsConstrained(c) **then**
  return DistanceBetween(C, c) {Case: 2}
else
  return SearchWidth(C, T, c, d) {Case: 3}
end if

with a valid path between these edges.

Finally, in Section 2.4 we show that the path found using this method given the radius of a unit is equivalent to the minimum length path between any points along these edges while avoiding obstacles. This will become a useful result later on.

2.1 Definitions

Consider the region which exists between the rays extending from vertex C toward A and from C toward B. Call this region R. Figure 6 illustrates this region for a triangle. We will say there is a valid path from edge a to edge b for a circular unit of radius r in triangle T if and only if there exists a path p such that:

1. One end of p is on edge a
2. The other end of p is on edge b
3. p is contained entirely in region R
4. For each point on p, there is no obstacle (vertex or point on a constrained edge) which is closer than r
5. For each point on p, there exists a point within distance r which is inside triangle T

Figure 10 shows an Arc Path for an example triangle.

The point 5 above warrants some discussion. It is possible to find a path possessing all the above properties except 5, however because the unit

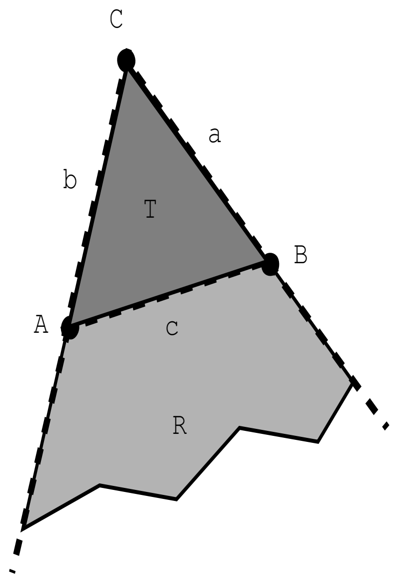

Figure 6: Region R for Triangle T when moving between Edges a and b

completely leaves the triangle T, we consider the path to instead go from edge a to edge c, continuing somehow through other triangles, and then returning to triangle T going from edge c to edge b.

The reason it is not required for the path to stay entirely within T is because sometimes a path might exist where the unit is always partially within T but the path itself might cross edge c. Suppose the triangle opposite edge c is T'. If we required the path to be entirely within T, such a path would require finding a path from a to c in T, from c to c in T', and then from c to b in T. This would complicate the problem unnecessarily and so such a path is only considered to be going through triangle T.

An example of a path p that leaves T but still satisfies this requirement is shown in Figure 7, and an example of a path that is invalid because it is $> r$ away from Triangle T at some points is shown in Figure 8.

Furthermore, we must make an exception to the rule for point 4. When the unit is crossing the edge a - that is, for points on the path for which some segment of length r extending from it intersects edge a - we do not consider obstacles in the region opposite edge a from region R. Similarly, when the unit crosses b, we do not consider obstacles in the region opposite edge b from R.

This is because the ultimate goal of finding these paths through individual triangles is to combine the paths through adjacent triangles together to form a single path through a triangulation. Thus obstacles on the opposite side of a will be the considered when finding a path through the triangle

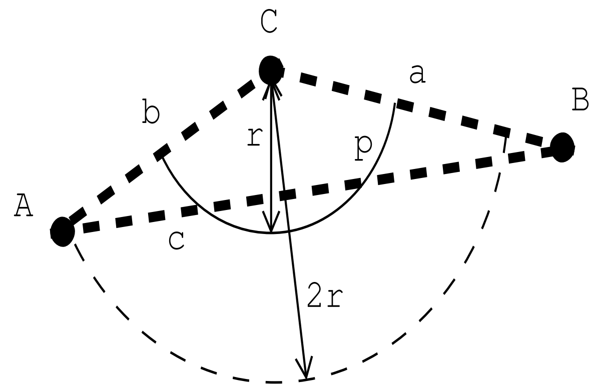

Figure 7: A valid path that is always within r of T

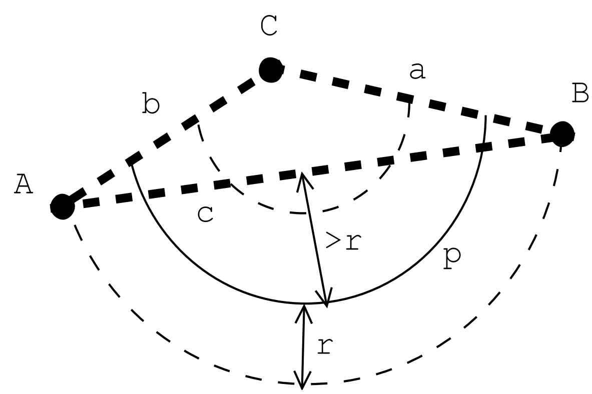

Figure 8: An invalid path that is $> r$ away from T at some points

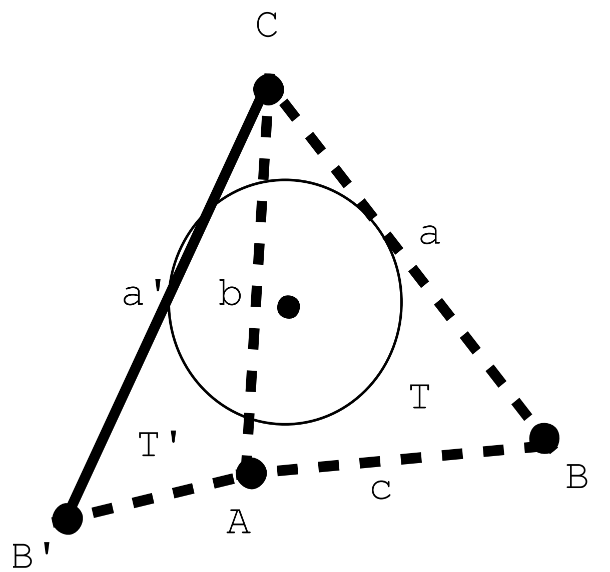

Figure 9: Obstacle outside of Region R interfering with a path inside of it

which shares edge a, and similarly for b. An example of such an obstacle interfering with a path outside of Region R is given in Figure 9.

With that being said, sometimes an obstacle outside of region R can interfere with a path through T. If, at some point along a path, the unit is crossing the boundary of R outside either edge a or edge b, then those obstacles are not being considered in the paths through the triangles sharing those edges and thus must be considered by T.

The Arc Method uses such paths which are described to "hug" the vertex joining edges a and b (vertex C). We consider such a path to be one where the unit is always touching this vertex throughout the path, hence it "hugs" vertex C.

That is, the unit's center follows a path that is always r away from vertex where r is the radius of the unit. In other words, the path is an arc at distance r from vertex C going from edge a to edge b. Also, as we will see, it is required that there are no obstacles within a similar arc of distance $2r$ from vertex C in order for the path to be clear.

For the purpose of using it in the proof of the Arc Method's completeness in Section 2.3, we will now show that an arc path will not cross the boundary of R other than through edges a and b. We will prove this by contradiction, assuming we have a valid path which crosses this boundary and showing it is not an arc path.

For a path hugging vertex C to be valid, the arc around C of radius $2r$ in region R must be free of obstacles. Because vertices are considered to be

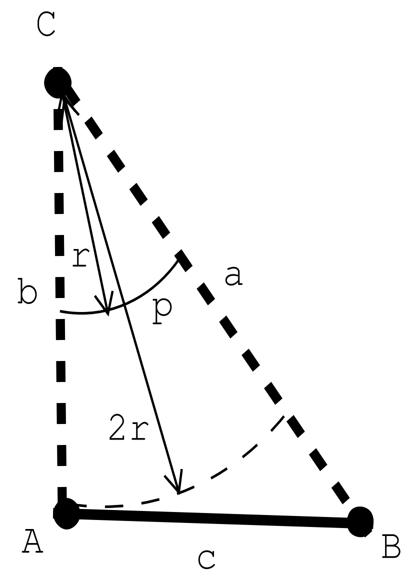

Figure 10: An example of an Arc Path

obstacles, both edges a and b must have length at least $2r$ in order for this path to be valid.

Without loss of generality, we will simply consider the boundary of R by edge b. The proof extends identically to the boundary by edge a. Now, the unit must cross the boundary of R not at edge b, and not partially at edge b since that would imply the unit overlaps vertex A, which is an obstacle, and the path would violate requirement 4 and be invalid.

Thus the closest point, which we will call w, on that boundary of R to some point on the path, which we will call p, must be outside of edge b. Thus since the segment from w to p must be perpendicular to the boundary of R, then p must be farther from vertex C than w. Also we know that w is farther from vertex C than vertex A, and vertex A is at least $2r$ from vertex C. Thus p would have to be farther than $2r$ from vertex C, which violates our definition that an arc path always be at distance r from vertex C. A diagram of this proof is found in Figure 11.

Thus an arc path will not cross a boundary of region R outside of edges a and b, as desired. By corollary, to verify requirement 4 of an arc path, we need only consider obstacles within region R.

2.2 Soundness

Here, we will prove that if there is no obstacle within $2r$ of vertex C in region R, then there is a path through triangle T from edge a to edge b hugging

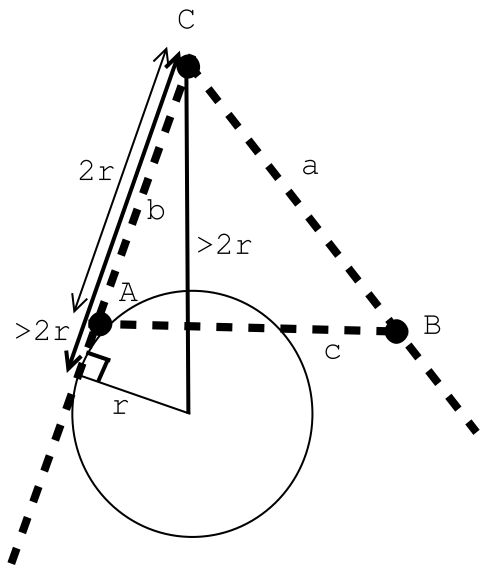

Figure 11: A unit crossing the boundary of Region R below Edge b

vertex C. By the very definition of these arc paths, this already satisfies requirements 1, 2, 3, and 5 in Section 2.1 above.

It remains to prove that if no obstacle is within distance $2r$ from vertex C - that is, there is no obstacle inside the arc of radius $2r$ around vertex C in region R - then the arc path satisfies requirement 4. This we will prove below.

Since the unit is travelling along the path hugging vertex C has radius r, every point on this path will be at distance r from C, by the definition of these arc paths. For requirement 4 not to be met, there must be an obstacle within distance of less than r from one of these points.

However, because of the triangle inequality, we know that an obstacle that is $\geq 2r$ from vertex C cannot be $< r$ from any point that is at distance r from that same point. This is depicted in Figure 12 for clarity. Thus, if there are no obstacles within distance $2r$ of vertex C in region R, there are none within distance r of any point along the path from edge a to b in triangle T hugging vertex C. Therefore, requirement 4 is met and the path is valid, and finally, the arc method is sound.

2.3 Completeness

In order to prove that if there exists a path from edge a to edge b in the triangle T for a unit of radius r then there are no obstacles within distance $2r$ of vertex C in region R, we will prove the contrapositive. Assume there

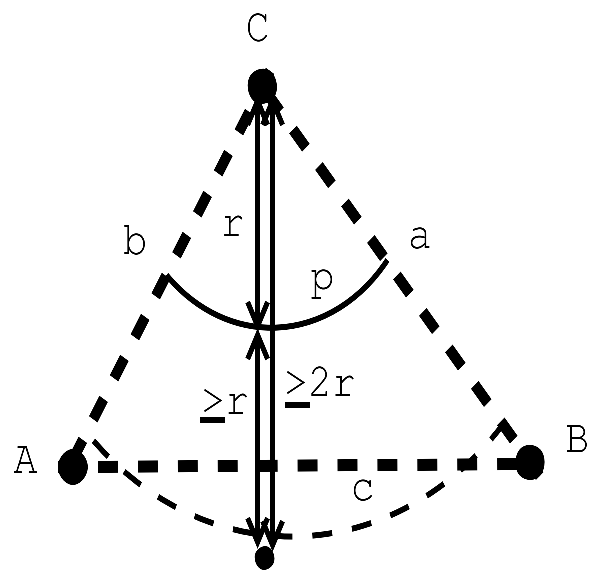

Figure 12: Using the triangle inequality to prove soundness

is an obstacle at distance $< 2r$ from vertex C in region R.

We will consider the point p to be the closest obstacle to vertex C in region R. Such a point must exist in region R because by definition (Section 2.1), only obstacles in region R can interfere with an arc path from edge a to edge b through triangle T.

Now consider the line segment going from vertex C to this point p. We know that the length of this line segment is $< 2r$. For the remainder of the proof, we will consider this point to be the only obstacle in region R. This is a relaxed constraint, and certainly, if no path exists with this single point obstacle, no path exists with any amount of obstacles including p.

Next, consider a segment between vertex C and p, and a ray extending from p perpendicular to and away from edge c of the triangle. One can see that these partition R into two sub-regions. This partitioning can be observed in Figure 13. Thus, we can see that any valid path travelling from edge a to edge b in region R must cross either the segment between C and p, or the ray extending from p, since edge a and edge b are in different sub-regions of R and a valid path must be entirely within R by requirement 3. We will cover both of these cases below.

First, assume the path of a unit with radius r crosses the segment between C and p at some point u. Since the segment between C and p has length $< 2r$, we know u must be at distance $< r$ from either vertex C or p (or both). Thus if the path crosses any point on this segment, it comes within distance r of some obstacle, and thus fails to meet requirement 4

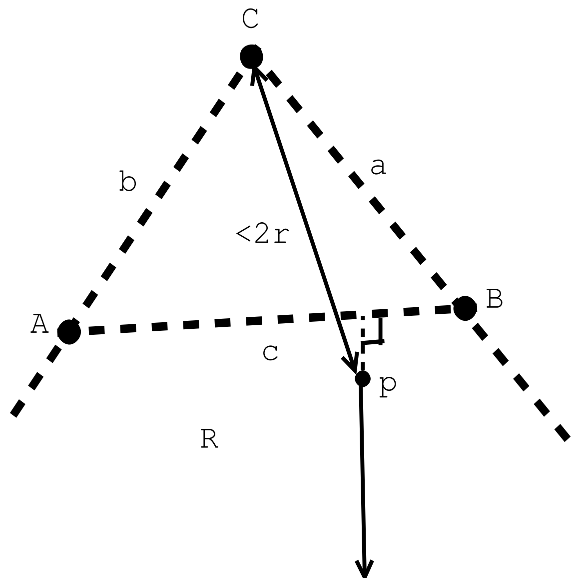

Figure 13: Partitioning of Region R into 2 sub-regions

above. This can be seen in Figure 14 where the above unit is interfering with vertex C (its centre is distance $< r$ from C) and the lower unit with point p.

Similarly, if the path crosses the ray extending from p perpendicular and away from edge c, it must cross at some point v on the ray that is at least distance r from p, otherwise it would fail to meet requirement 4 and would be an invalid path. Figure 15 shows that a unit crossing this ray is indeed at distance $> r$ from triangle T. We also use that there cannot be an obstacle within a triangle in a triangulation: obstacles are either represented as vertices or edges of disjoint triangles.

Thus the point v where the path crosses the ray must be at least r away from p. Since the ray is perpendicular to c, the closest edge of T to p, the point that is distance r away from v is along that ray. Since v is at least r away from p, which itself is outside of T, it follows that at point v, there is no point within distance r which is inside triangle T. Thus requirement 5 is violated and the path is invalid.

Finally we conclude that since there is nowhere that a path from edge a can cross the partition to edge b and remain valid, all such paths are invalid. Thus if a valid path exists for moving a circular unit of radius r from edge a to edge b in triangle T, there must be no obstacles within distance $2r$ of vertex C in region R, as desired.

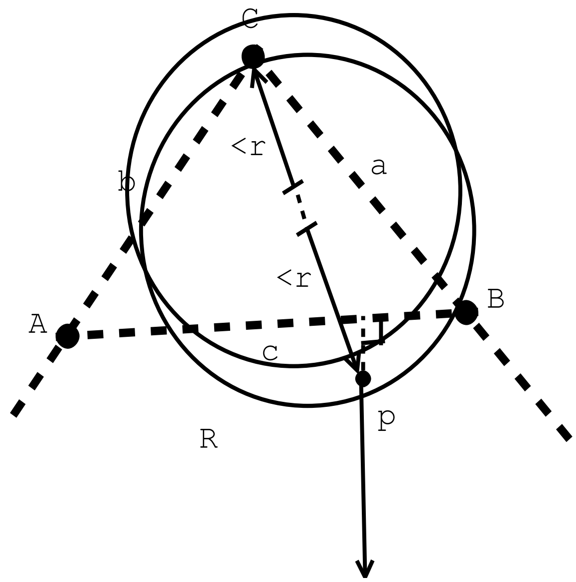

Figure 14: A unit trying to pass between C and p

2.4 Minimal Length

The proof that the arc path between two edges of a triangle is indeed the shortest valid path between those edges will go here once finished.

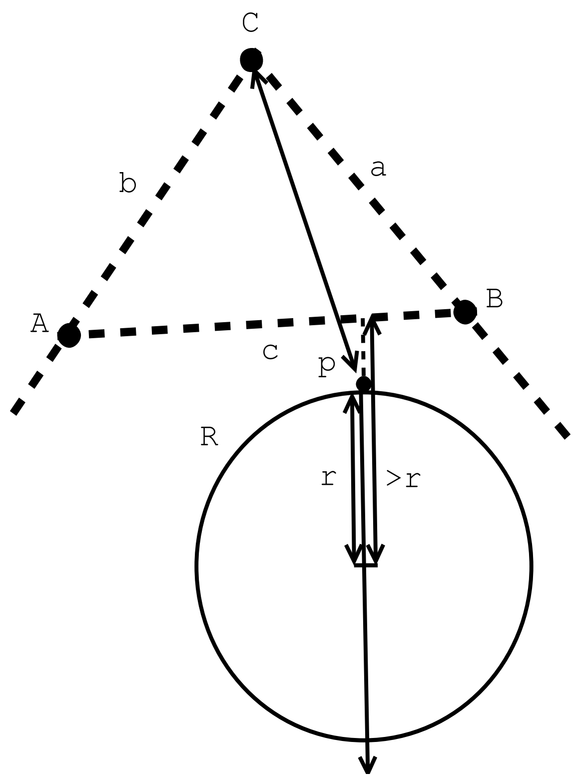

Figure 15: A unit trying to pass below point p