

# 1 The Randomized Block Design

When introducing ANOVA, we mentioned that this model will allow us to include more than one categorical factor(explanatory) or confounding variables in the model.

In a first step we will now include a block variable (factor). This is usually considered a variable that is a confounding variable, i.e. not of interest by itself but has an influence on the response variable and should for this reason be included.

Sometimes a study is designed to include such a variable in order to reduce the variability in the response variable and therefore to require a smaller sample size. Generally each treatment is used exactly once within each block, in conclusion:

if we have  $k$  treatments and  $b$  block, then the total sample size is  $n = b \cdot k$ .

The concept origins from agricultural studies, when studying yields of certain grain, e.g. grain under different conditions.

## Example 1

(Yield and Early Growth Responses to Starter Fertilizer in No-Till Corn Assessed with Precision Agriculture Technologies, Manuel Bermudez and Antonio P. Mallarino (2002)) Several trials were conducted in the 1990's to evaluate corn yield and early growth responses to starter fertilizer in Iowa farmers' fields that had 8 to 14 yr of no-till management. Soil series represented in the experimental areas varied across fields and were among typical agricultural soil series of Iowa and neighboring states.

To illustrate, assume three different starters were used. In order to limit the number of field that needed to be planted, 10 locations were chosen, where each field was divided into three parts, then each part was treated with a different starter (randomly assigning the treatment to each part). Beside the fertilizer all other agricultural practises were to be the same.

For this example:

response variable=yield kg/m<sup>2</sup>

treatment(factor) variable= starter (A, B, C)

block(confounding) variable=loacation (1,2,3,4,5,6,7,8,9,10)

## Example 2

The cutting speeds of four types of tools are being compared in an experiment. Five cutting materials of varying degree of hardness are to be used as experimental blocks. The data giving the measurement of cutting time in seconds appear in the table below

Treatment	Block					Treatment Means
	1	2	3	4	5	
1	12	2	8	1	7	$\bar{x}_{T1}=6$
2	20	14	17	12	17	$\bar{x}_{T2}=16$
3	13	7	13	8	14	$\bar{x}_{T3}=11$
4	11	5	10	3	6	$\bar{x}_4=7$
Block Means	$\bar{x}_{B1}=14$	$\bar{x}_{B2}=7$	$\bar{x}_{B3}=12$	$\bar{x}_{B4}=6$	$\bar{x}_{B5}=11$	$\bar{x}=10$

## The Randomized Block Design Model

$x_{ij}$  = the measurement for treatment  $i$  in block  $j$  (remember there is precisely one such measurement)  
then

$$x_{ij} = \mu + \alpha_i + \beta_j + e_{ij}$$

where:  $\mu$  = overall mean

$\alpha_i$  = effect of treatment  $i$  (difference with  $\mu$ )

$\beta_j$  = effect of block  $j$  (difference with  $\mu$ )

$e_{ij}$  = error in measurement for treatment  $i$  and block  $j$ .

A positive value for  $\alpha_i$  indicates that the mean of the response variable is greater than the overall mean for treatment  $i$ .

Assume  $e_{ij} \sim \mathcal{N}(0, \sigma)$  for all measurements.

In the analysis of variance instead of only explaining the variance through error and treatment, we also include the block as a possible source for variance in the data. For that reason we now also include

Sum of Squares for Block ( $SSB$ ) with our analysis:

- Sum of Squares for treatment:

$$SST = \sum_{i=1}^k b(\bar{x}_{Ti} - \bar{x})^2, df_T = k - 1$$

- Sum of Squares for block:

$$SSB = \sum_{j=1}^b k(\bar{x}_{Bj} - \bar{x})^2, df_B = b - 1$$

- Total Sum of Squares:

$$TotalSS = \sum_{i,j} (x_{ij} - \bar{x})^2, df_{Total} = n - 1$$

- Sum of Squares for error:

$$SSE = TotalSS - SST - SSB, df_E = n - k - b + 1$$

Summarized in an ANOVA-table:

**ANOVA Table for a Randomized Block Design**

Source	$df$	$SS$	$MS$	$F$
Treatments	$k - 1$	$SST$	$MST = SST/(k - 1)$	$MST/MSE$
Blocks	$b - 1$	$SSB$	$MSB = SSB/(b - 1)$	$MSB/MSE$
Error	$n - k - b + 1$	$SSE$	$MSE = SSE/(n - k - b + 1)$	
Total	$n - 1$	$TotalSS$		

### Example 3

Let us find the ANOVA table for the cutting example:

- Sum of Squares for treatment:

$$SST = \sum_{i=1}^k b(\bar{x}_{Ti} - \bar{x})^2 = 5(6-10)^2 + 5(16-10)^2 + 5(11-10)^2 + 5(7-10)^2 = 310, df_T = k-1 = 3$$

- Sum of Squares for block:

$$SSB = \sum_{j=1}^b k(\bar{x}_{Bj} - \bar{x})^2 = 4(14-10)^2 + \dots + 4(11-10)^2 = 184, df_B = b-1 = 4$$

- Total Sum of Squares:

$$TotalSS = \sum_{i,j} (x_{ij} - \bar{x})^2 = (12-10)^2 + (2-10)^2 + \dots + (6-10)^2 = 518, df_{Total} = n-1 = 5(4)-1 = 19$$

- Sum of Squares for error:

$$SSE = TotalSS - SST - SSB = 518 - 310 - 184 = 24, df_E = n - b - k + 1 = 12$$

### ANOVA Table for a Randomized Block Design

Source	<i>df</i>	<i>SS</i>	<i>MS</i>	<i>F</i>
tool	3	310	$MST = 103.3$	51.7
material	4	184	$MSB = 46$	23
Error	12	24	$MSE = 2$	
Total	19	518		

Based on the statistics in the ANOVA table, we can now test for a treatment effect and for a block effect.

### The ANOVA F-Test(Randomized Block Design)

1. The Hypotheses are

$$H_0 : \alpha_1 = \alpha_2 = \dots = \alpha_k = 0 \text{ versus}$$

$$H_a : \text{at least one of the values differs from the others}$$

2. Assumption: The measurements represent random samples from  $k$  treatments and  $b$  blocks.

The error is normally distributed with mean 0 and equal standard deviation,  $\sigma$  for all  $bk$  combinations of treatments and blocks.

(We could also say: For each treatment/block combination the response variable is normally distributed with the same standard deviation  $\sigma$ .)

3. Test statistic:

$$F_0 = \frac{MST}{MSE}$$

based on  $df_1 = (k-1)$  and  $df_2 = (n - k - b + 1)$ .

4. P-value:  $P(F > F_0)$ , where  $F$  follows an  $F$ -distribution with  $df_1 = (k - 1)$  and  $df_2 = (n - k - b + 1)$ .
5. Decision: If P-value  $\leq \alpha$ , then reject  $H_0$ .  
If P-value  $> \alpha$ , then do not reject  $H_0$ .
6. Put into context.

In order to test for a difference in the blocks,

1. The Hypotheses are

$$H_0 : \beta_1 = \beta_2 = \dots = \beta_b = 0 \text{ versus} \\ H_a : \text{at least one of the values differs from the others}$$

2. Assumption: same as above
3. Test statistic:

$$F_0 = \frac{MSB}{MSE}$$

based on  $df_1 = (b - 1)$  and  $df_2 = (n - k - b + 1)$ .

4. P-value:  $P(F > F_0)$ , where  $F$  follows an  $F$ -distribution with  $df_1 = (b - 1)$  and  $df_2 = (n - k - b + 1)$ .
5. Decision: same as above
6. Put into context.

Once we determined with the F test that a difference between the treatment means exist, we will use a multiple comparison analysis, to determine where the differences occur.

Use Bonferroni

The CIs for the pairwise comparisons are the same like for the one-way ANOVA, only change are the degrees of freedom

Let  $\gamma = C_1\mu_{T1} + C_2\mu_{T2} + \dots + C_k\mu_{Tk}$  a contrast with sample contrast  $c = C_1\bar{x}_{T1} + C_2\bar{x}_{T2} + \dots + C_k\bar{x}_{Tk}$ .

$(1 - \alpha)100\%$  **Confidence Interval for  $\gamma$**

$$c \pm t_{1-\alpha/2}^{df} SE(c)$$

$t_{1-\alpha/2}^{df}$  is the  $1 - \alpha/2$  percentile of the t-distribution with  $df = n - k - b + 1$ .

### Continue Example:

We test that the mean cutting speed is independent from the tool used, that is

1. The Hypotheses are

$$H_0 : \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0 \text{ versus} \\ H_a : \text{at least one of the values differs from the others}$$

$$\alpha = 0.05$$

- Assumption: The samples were independent cuts, so the samples are independent. (the operator of the machines made all cuts in a random order, and refreshed often enough to have no tiring effect)

Have to check if there is any doubt that the error follows a normal distribution and equal standard deviation for all  $bk = 20$  combinations of treatments and blocks.

See Residual Analysis below.

- Test statistic:

$$F_0 = \frac{MST}{MSE} = 51.7$$

based on  $df_1 = 3$  and  $df_2 = 20 - 4 - 5 + 1 = 12$ .

- P-value:  $P(F > F_0) < 0.005$ , using table IX (since  $51.7 > 7.23$ ).
- Decision: Since P-value  $< \alpha = 0.05$  reject  $H_0$ .
- At significance level of 5% the data provide sufficient evidence that the mean cutting speeds are not the same for the four tools.

Use Bonferroni, to determine where the differences are.

$m = k(k - 1)/2 = 6$  use  $\alpha^* = \alpha/m = 0.05/6 = 0.0083$  To find the critical value, we need  $\alpha^*/2 = 0.0041 \approx 0.005$  (closest in the table), for  $df = 12$ , we get a critical value of  $t_{0.005}^{12} = 3.055$ .

Since the sample size is the same for all treatments the Margin of Error will be the same for all pairwise comparisons

$$ME = t_{0.005}^{12} \cdot \sqrt{MSE} \sqrt{\frac{1}{b} + \frac{1}{b}} = 3.055 \cdot \sqrt{2} \cdot \sqrt{2/5} = 2.73$$

Now rank the sample means and underline those, which differ less than 2.73

means	<u>6</u>	<u>7</u>	11	16
tools	T1	T4	T2	T3

With an experiment wise error rate of 0.05 the data provide sufficient evidence that the mean cutting speed for tools 1 and 4 are significantly different from the mean cutting speed for tool 2 as well as tool 3, further the mean cutting speed for tools 2 and 3 are significantly different. Mean cutting speeds for tools 1 and 4 are not significantly different.

Also, the mean cutting speeds for tools 1 and 4 are significantly shorter than for the two other tools.

A test for a difference in the means of the blocks is of no interest, because the materials tested are not relevant, they were only helpful for testing the tools for a variety of materials.

## Residual Analysis

Residuals in a RBD are calculated by subtracting the estimates for the treatment and block effects from each measurement:

$$r_{ij} = x_{ij} - \alpha_i - \beta_j$$

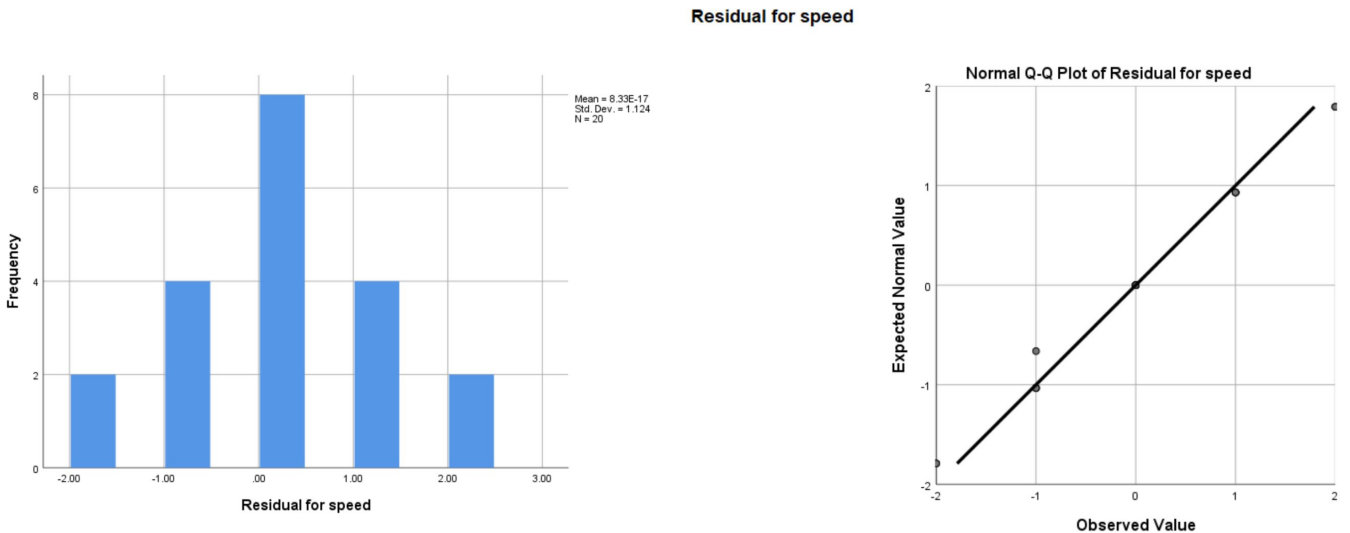
- Create a histogram and a QQ-plot for the residuals to check for normality

2. Create side-by-side box plots for the residuals, one for the treatments and one for the blocks, to check for homoscedasticity.

## Continue Example

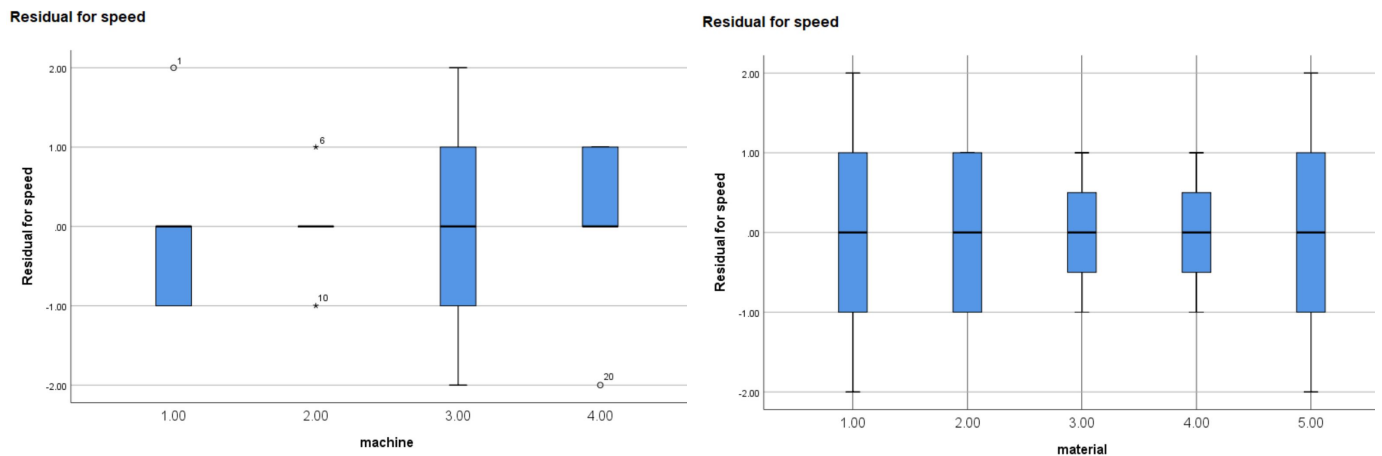
Residual Analysis for the RBD model fitting machine and material effects on cutting speed.

1. Checking normality of the error



Observing histogram and QQ-plot, the residuals do not show a strong deviation from plots that would be expected from normally distributed data, so no evidence against the assumption are noted

2. Checking homoscedasticity



The boxplot of the residuals for material show even sized boxes and tails and does not raise any concerns for a violation of the homoscedasticity assumption.

The boxplot of the residuals for machine show very different sized boxes and tails and do raise concerns for a violation of the homoscedasticity assumption.

The standard deviations of the residuals are 1.22, 0.71, 1.58, and 1.22, for the four machines, respectively. Violating the rule of thumbs, that no standard deviation should be more than

twice another standard deviation. Given the small sample size, and that there is only one small concern, we can still trust the analysis.