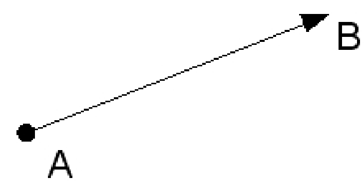
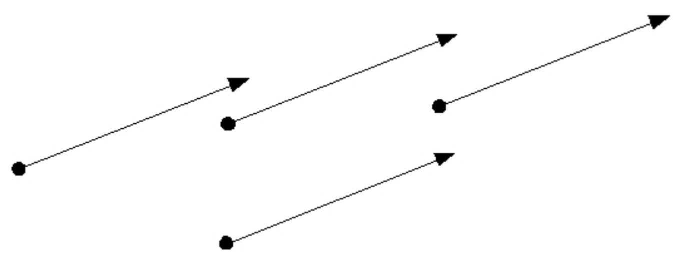
1 Vectors in 2-Space and 3-Space

1.1 Geometric Introduction

Vectors are represented geometrically as line segments with a direction (or arrows) in 2-space or 3-space.
The length of the vector describes its magnitude and the direction of the arrow determines the direction.
Use lowercase bold face letter to represent vectors.

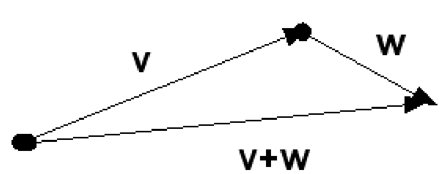


In the figure above A is the initial point and B is the terminal point of the vector $\mathbf{v} = \overrightarrow{AB}$.
Vectors of the same length and direction are called equivalent.



For equivalent vectors $\mathbf{v}$ and $\mathbf{w}$ we write $\mathbf{v} = \mathbf{w}$.

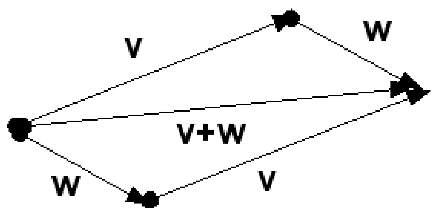
Definition 1
If $\mathbf{v}$ and $\mathbf{w}$ are any vectors than the sum, $\mathbf{v} + \mathbf{w}$, is the vector that has the same initial point as $\mathbf{v}$ and the same final point as $\mathbf{w}$, when the initial point of $\mathbf{w}$ coincides with the final point of $\mathbf{v}$



Theorem 1
For any vectors $\mathbf{v}$ and $\mathbf{w}$

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$$

the sum is commutative.



Definition 2

(a) The vector of length 0 is the zero vector, $\mathbf{0}$.

(b) For any vector $\mathbf{v}$

$$\mathbf{0}+\mathbf{v}=\mathbf{v}+\mathbf{0}=\mathbf{v}$$

(c) For any non zero vector $\mathbf{v}$ the negative of $\mathbf{v}$, $-\mathbf{v}$, is defined to be the vector of the same size as $\mathbf{v}$ but is oppositely directed.

(d) $-\mathbf{0}=\mathbf{0}$

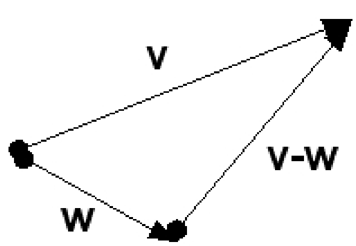
Theorem 2
For any vector $\mathbf{v}$

$$\mathbf{v}+(-\mathbf{v})=\mathbf{0}$$

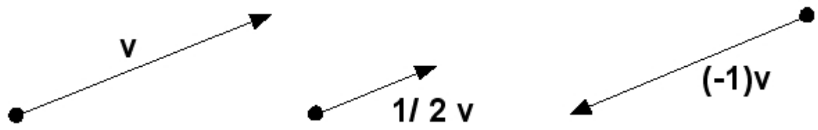
Definition 3
For any vectors $\mathbf{v}$ and $\mathbf{w}$, the difference is

$$\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$$

To obtain $\mathbf{v}-\mathbf{w}$ from $\mathbf{v}$ and $\mathbf{w}$ with out $-\mathbf{w}$, then the vector with initial point at terminal point of $\mathbf{w}$ and terminal point being the terminal point of $\mathbf{v}$ is $\mathbf{v}-\mathbf{w}$



Definition 4
If $\mathbf{v}$ is a nonzero vector and $k \in \mathbb{R}\backslash\{0\}$, then the product $k\mathbf{v}$ is defined to be the vector that has $|k|$ times the length of $\mathbf{v}$ and the same direction as $\mathbf{v}$, if $k>0$ and the opposite direction if $k<0$.
If $k=0$ or $\mathbf{v}=\mathbf{0}$, then define $k\mathbf{v}=\mathbf{0}$, the zero vector.



1.2 Vector Computation with Components

Vectors can be easily visualized in the plane or in 3-dimensional space.
The components of a vector are the coordinates of the terminal point of the vector when the initial point falls at the origin of a rectangular coordinate system.

Write $=(v_1, v_2)$.

Two vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ are equivalent iff

$$v_1 = w_1 \text{ and } v_2 = w_2$$

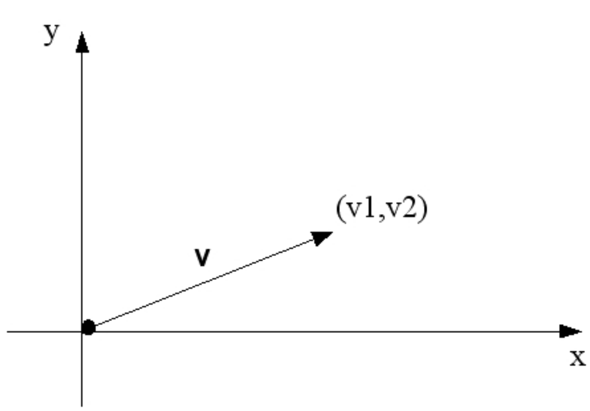
Theorem 3

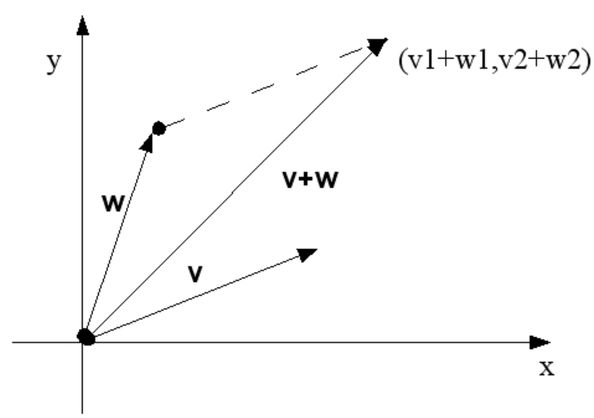
For two vectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$ the components of the sum are

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$$

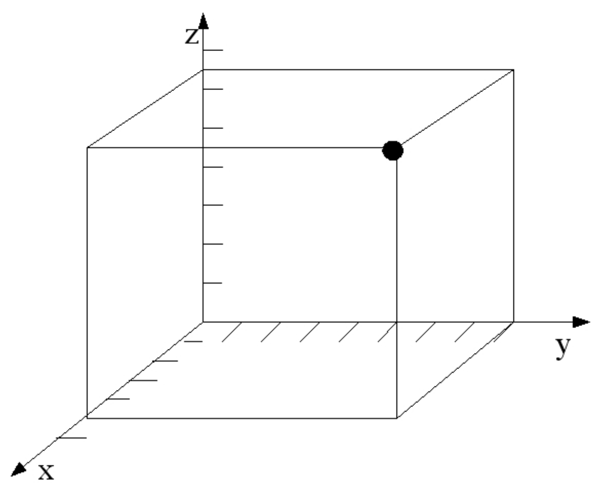
For a vectors $\mathbf{v} = (v_1, v_2)$ and $k \in \mathbb{R}$ the components of the product are

$$k\mathbf{v} = (kv_1, kv_2)$$

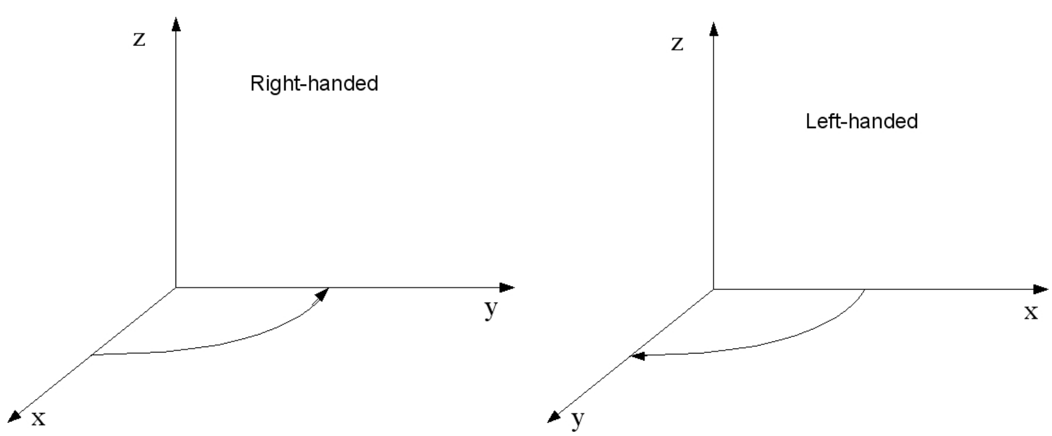




3-space
Vectors in three space can be described by triples of real number, indicating their terminal points in a 3 dimensional coordinate system, assuming their initial point is in the origin, O, of the coordinate system.
After choosing an origin,O, the coordinate axes are three mutually perpendicular lines through O, they are labeled x, y and z and include a unit of length.
Each pair of coordinate axes determines a coordinate plane, they are called xy-, xz-, and yz-plane.
Each point P is characterized by three numbers (x, y, z), the coordinates.

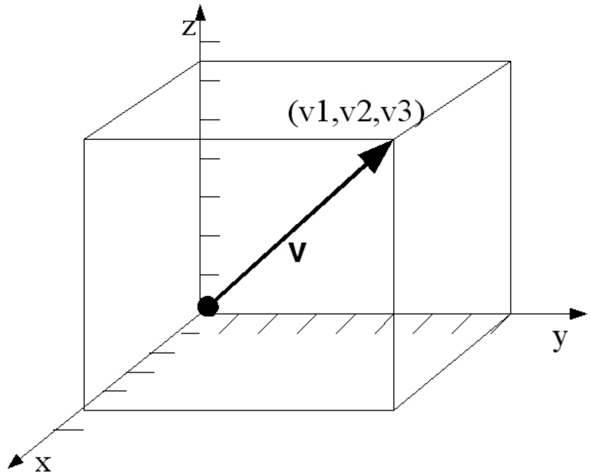


The point in the diagram is (5, 8, 6.5)
Rectangular coordinate systems are either left handed or right handed.



Here we will only use right handed coordinate systems.
The initial point of a vector $\mathbf{v}$ in 3-space is in the origin and the coordinates of the terminal points are the components.

$$\mathbf{v} = (v_1, v_2, v_3)$$



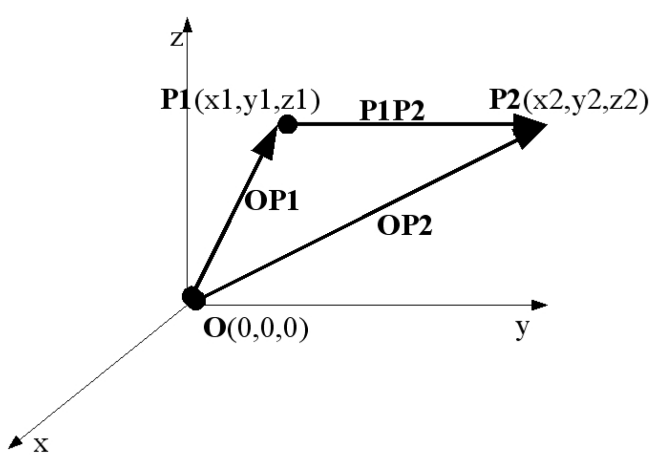
Definition 5
For vectors $\mathbf{v}, \mathbf{w}$ in 3-space

1. $\mathbf{v}$ and $\mathbf{w}$ are equivalent, iff $v_1 = w_1, v_2 = w_2, v_3 = w_3$
2. $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, v_3 + w_3)$
3. $k\mathbf{v} = (kv_1, kv_2, kv_3)$, for $k \in \mathbb{R}$

Example 1
If $\mathbf{v} = (-1, -2, -3)$ and $\mathbf{w} = (3, 2, 7)$ then
$\mathbf{v} + \mathbf{w} = (2, 0, 4)$, $2\mathbf{v} = (-2, -4, -6)$, $-\mathbf{w} = (-3, -2, -7)$, and $\mathbf{v} - \mathbf{w} = (-4, -4, -10)$

If a vector $\overrightarrow{P_1P_2}$ is not being positioned in the origin and has initial point $P_1 = (x_1, y_1, z_1)$ and terminal point $P_2 = (x_2, y_2, z_3)$, then

$$\overrightarrow{P_1P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

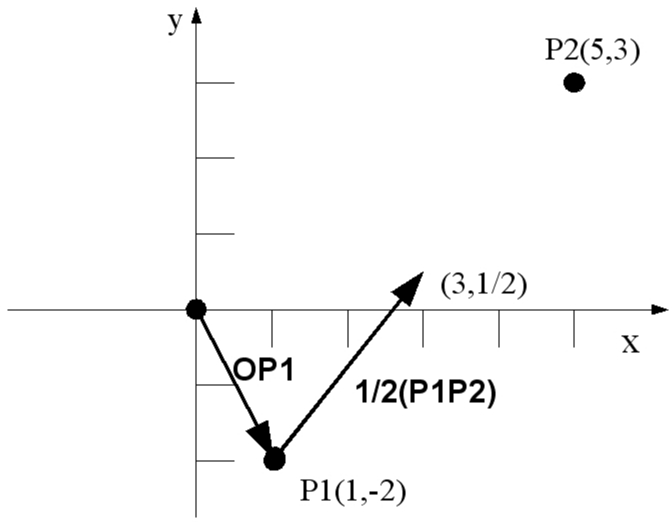


$\overrightarrow{OP_1} = (x_1, y_1, z_1), \overrightarrow{OP_2} = (x_2, y_2, z_2)$, and $\overrightarrow{P_1P_2} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is the difference between terminal point and initial point.

Example 2

1. $\mathbf{v}$ is the vector with initial point $(1,-2,3)$ and end point $(2,-3,4)$, then $\mathbf{v}=(2-1,-3-(-2),4-3)=(1,-1,1)$.
2. The midpoint between $P_1=(1,-2)$ and $P_2=(5,3)$ is

$$\overrightarrow{OP_1}+\frac{1}{2}\overrightarrow{P_1P_2}=(1,-2)+\frac{1}{2}(5-1,3-(-2))=(1+2,-2+\frac{5}{2})=(3,\frac{1}{2})$$



3. $\mathbf{v}=(1,-1,1),\mathbf{w}=(3,-2,1)$, then

$$5\mathbf{v}-2\mathbf{w}=(5(1)-2(3),5(-1)-2(-2),5(1)-2(1))=(-1,-1,3)$$

1.3 Properties of vector arithmetic and the norm of a vector

After observing that the vector operations are the same as the matrix operations the following theorem is a consequence from the results on matrix operations

Theorem 4
If $\mathbf{u},\mathbf{v}$, and $\mathbf{w}$ are vectors and $k,l\in\mathbb{R}$, then

1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$
2. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$
3. $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$
4. $\mathbf{u}+-\mathbf{u}=-\mathbf{u}+\mathbf{u}=\mathbf{0}$
5. $k(l\mathbf{u})=(kl)\mathbf{u}$
6. $k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\mathbf{v}$
7. $(k+l)\mathbf{u}=k\mathbf{u}+l\mathbf{u}$

8. $1\mathbf{u} = \mathbf{u}$

The length of a vector is a characterizing property, it is called its norm.

Theorem 5

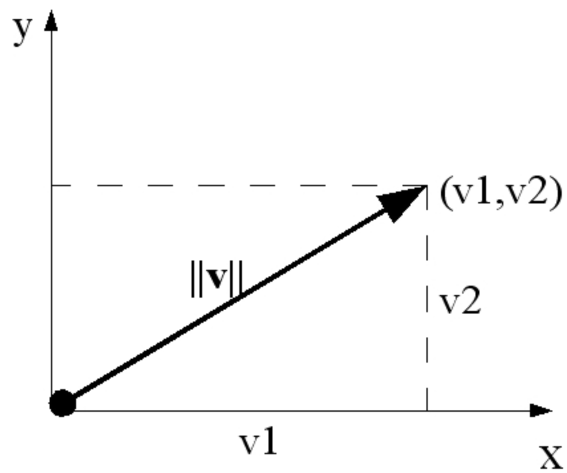
The norm of a vector $\mathbf{v} = (v_1, v_2)$ in 2- space is

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$$

The norm of a vector $\mathbf{v} = (v_1, v_2, v3)$ in 3- space is

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

Proof: Use Theorem of Pythagoras (for a rectangular triangle $z^2 = x^2 + y^2$)



then

$$||\mathbf{v}||^2 = v_1^2 + v_2^2 \Leftrightarrow ||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2}$$

Theorem 6
If $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ are points in 3-space , then the distance d between them is the norm of the vector $\overrightarrow{P_1P_2}$. and

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_3)^2}$$

Similar for two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in 2-space, then the distance d between them is the norm of the vector $\overrightarrow{P_1P_2}$. and

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 3
Let $\mathbf{v} = (5, 2, -1)$, then $||\mathbf{v}|| = \sqrt{25 + 4 + 1} = \sqrt{30}$.
The distance between $P_1 = (0, 0, 1)$ and $P_2 = (1, 1, 1)$ is $d = \sqrt{(-1)^2 + (-1)^2 + 0^2} = \sqrt{2}$

Remark: For a vector $\mathbf{u}$ and $k \in \mathbb{R}$

$$||k\mathbf{u}|| = |k| \cdot ||\mathbf{u}||$$