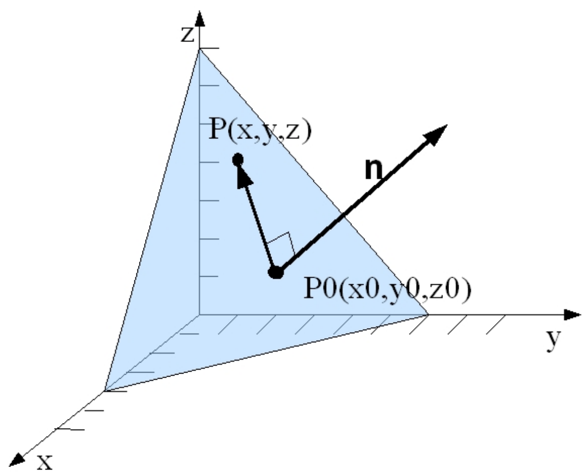
0.1 Lines and Planes in 3-space

Planes in 3-space

In Geometry a line in 2-space can be identified through its slope and one of its points.
Similarly one can specify a plane in 3-space by giving its inclination and one of its points. For indicating the inclination it is convenient to report a vector which is orthogonal to the plane. Such a vector is called a **normal**.

Find the equation of a plane with point $P_0(x_0, y_0, z_0)$ and normal $\mathbf{n} = (a, b, c)$.



Have a look at the diagram, and observe that the plane includes all those points $P(x, y, z)$, where $\overrightarrow{P_0P}$ is orthogonal to $\mathbf{n}$, that is

$$\overrightarrow{P_0P} \cdot \mathbf{n} = 0$$

Since $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ one can rewrite the equation as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This equation is called the *point-normal form* of the equation of a plane.

Example 1

Find the equation of the plane through point $P_0(1, -1, 6)$ and normal $\mathbf{n} = (4, 2, -1)$

$$4(x - 1) + 2(y - 2) - (z - 6) = 0 \Leftrightarrow 4x + 2y - z - 2 = 0$$

In general one can rewrite the point normal form as

$$ax + by + cz + d = 0$$

This is called the *general form* of the equation of a plane.

Theorem 1
If $a, b, c, d \in \mathbb{R}$ and $a, b, c,$ are not all zero, then the graph of the equation

$$ax + by + cz + d = 0$$

is a plane in 3-space having $\mathbf{n} = (a, b, c)$ as normal.

Intersection of three planes

The solution of a linear system consisting of three equations in three variables can be interpreted, as the point of intersection of three planes
The solutions of the linear system

$$\begin{aligned} ax + by + cz &= k_1 \\ dx + ey + fz &= k_2 \\ gx + hy + iz &= k_3 \end{aligned}$$

represent the points of intersection of the planes given by the three equations.

Example 2
To find the general form of a plane through three points $P_1(1,2,3), P_2(0,2,-1)$ and $P_3(3,4,5)$. either find the solution of the linear system

$$\begin{aligned} a + 2b + 3c + d &= 0 \\ 2b - c + d &= 0 \\ 3a + 4b + 5c + d &= 0 \end{aligned}$$

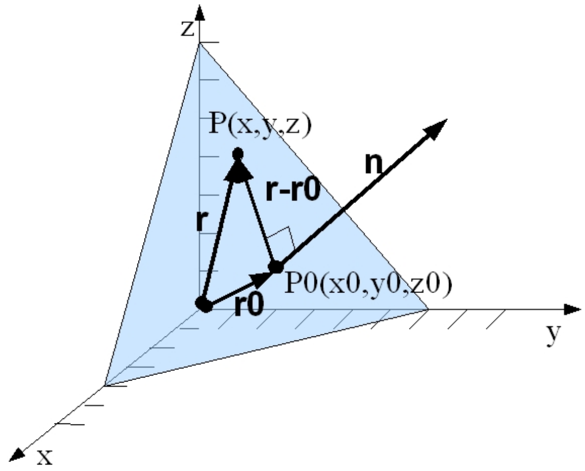
(all three point have to satisfy the general form $ax+by+cz+d=0$), or find the point-normal form.
Vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ are parallel to the plane for that reason their cross product must be orthogonal to the plane and be a normal.

$$\mathbf{n} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = (-1, 0, -4) \times (2, 2, 2) = (8, 6, -2)$$

We also know that P_1 is a point on the plane so a point normal form for the plane is

$$8(x-1) + 6(y-2) - 2(z-3) = 0$$

Vector form of the equation of a plane
An alternative form of writing the equation of a plane can be obtained by using vectors.



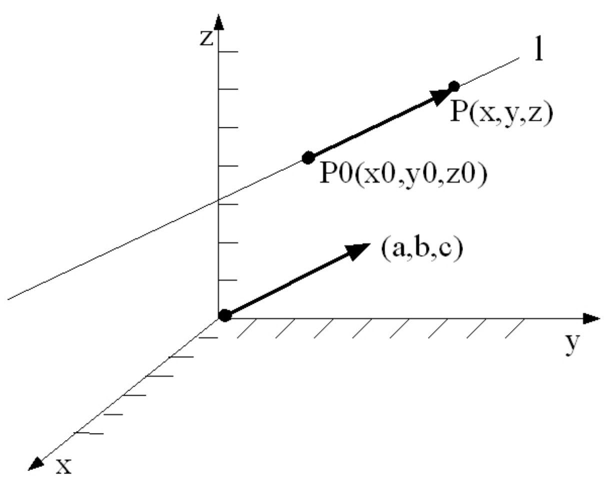
If $\mathbf{r_0} = (x_0, y_0, z_0)$ is the vector from the origin to point P_0 then each point $P(x, y, z)$ on the plane with vector $\mathbf{r} = (x, y, z)$ from the origin to $P(x, y, z)$ is characterized by the property that the vector $\mathbf{r} - \mathbf{r_0}$ is orthogonal to the normal $\mathbf{n}$, that is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

This is called the *vector form* of the equation of a plane.
For the example above this would lead to

$$(8, 6, -2) \cdot (\mathbf{r} - (1, 2, 3)) = 0$$

Lines in 3-space
Suppose that l is the line through point $P_0(x_0, y_0, z_0)$ and parallel to vector $\mathbf{v} = (a, b, c)$.



Then $P(x, y, z)$ is on l if and only if the vector $\overrightarrow{P_0P}$ is parallel to $\mathbf{v}$. That is it exists a number in $t \in \mathbb{R}$ so that

$$\overrightarrow{P_0P} = t\mathbf{v} \Leftrightarrow (x - x_0, y - y_0, z - z_0) = (ta, tb, tc)$$

or

$$x - x_0 = at, y - y_0 = bt, z - z_0 = ct \Leftrightarrow x = x_0 + at, y = y_0 + bt, z = z_0 + ct$$

The *parametric equations* for line l are then given by

$$x = x_0 + at, y = y_0 + tb, z = z_0 + tc, \quad t \in \mathbb{R}$$

Example 3
The line through point $(1, -2, 4)$ parallel to $\mathbf{v} = (1, 1, -1)$ has parametric equations

$$x = 1 + t, y = -2 + t, z = 4 - t, \quad t \in \mathbb{R}$$

Example 4
Intersection of a line and the xy-Plane
Where does the line from the example above intersect the xy-plane? The $xy-$plane consists of all point with third component being 0.
Therefore the point on the line with third component being 0, has to satisfy $0 = z = 4 - t \Leftrightarrow t = 4$, therefore using the parametric equations of the line with $t = 4$ yields that the point on the line in the xy-plane has coordinates

$$(1 + 4, -2 + 4, 0) = (5, 2, 0)$$

Example 5
Line of intersection of two planes
Find the parametric equations of the line of intersection of the planes given by

$$x + 2y - 1z + 1 = 0, \quad 2x - y + 3z - 4 = 0$$

All points on the line of intersection have to satisfy both equations. To determine the points we can solve the linear system

$$\begin{aligned} x + 2y - 1z &= -1 \\ 2x - y + 3z &= 4 \end{aligned}$$

In this case Gaussian elimination leads to the solutions for $t \in \mathbb{R}$: $z = t, y = -6/5 + t, x = 3/5 - t$.
The parametric equations for the line of intersection are

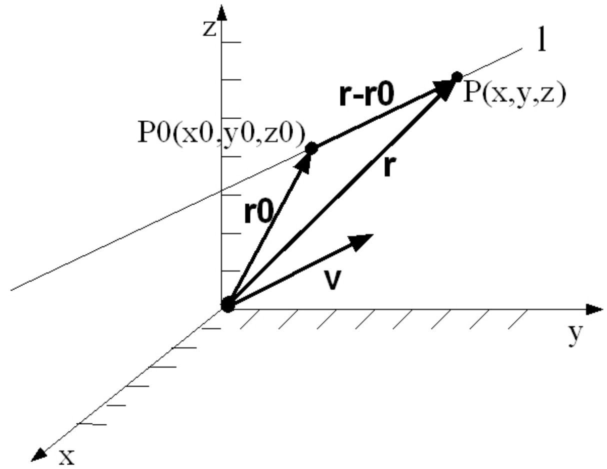
$$x = 3/5 - t, y = -6/5 + t, z = t, \quad t \in \mathbb{R}$$

Vector form of the equation of a line
Find the line through a given point P_0 and parallel to a given vector $\mathbf{v}$
If $\mathbf{r_0} = (x_0, y_0, z_0)$ is the vector from the origin to the given point P_0 on the line, then each point $P(x, y, z)$ on the line with $\mathbf{r} = (x, y, z)$ being the vector from the origin to a point $P(x, y, z)$ is characterized by the property that the vector $\mathbf{r} - \mathbf{r_0}$ is parallel to the given vector $\mathbf{v}$, the line is supposed to be parallel to.
Therefore for all points $P(x, y, z)$ on the line, there exists a $t \in \mathbb{R}$ so that

$$(\mathbf{r} - \mathbf{r_0}) = t\mathbf{v} \Leftrightarrow \mathbf{r} = \mathbf{r_0} + t\mathbf{v}$$

The *vector form* of the equation of a line is given by

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}, \quad t \in \mathbb{R}$$



Example 6
The equation

$$(1,2,3)+t(-1,2,4), \quad t\in\mathbb{R}$$

is the vector equation of the line through $P_0(1,2,3)$ parallel to $\mathbf{v}=(-1,2,4)$

Distance Problems

- The distance D between a point $P_0(x_0,y_0,z_0)$ and a plane given by $ax+by+cz+d=0$ equals

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$$

  The proof is similar to the proof for the distance of a point and a line in 2-space.

Example 7
Distance between two parallel planes
The planes given by

$$2x-3y+z-1=0 \text{ and } 4x-6y+2z+4=0$$

are parallel, because the normal vectors of the two planes are parallel ($\mathbf{n_1}=(2,-3,1),\mathbf{n_2}=(4,-6,2)$).
To find the distance between the planes choose an arbitrary point on one plane and then find the distance using the formula above. To choose a point one may choose $x=0,y=0$ and find that point $(0,0,1)$ is on the first plane.
Therefore the distance between the two planes is

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|4(0)-6(0)+2(1)+4|}{\sqrt{4^2+(-6)^2+2^2}}=\frac{6}{\sqrt{56}}$$