Subspaces of $\mathbb{R}^n$

Def. $S \subset \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if

(1) $\vec{x}, \vec{y} \in S \Rightarrow \vec{x} + \vec{y} \in S$

(2) $a \in \mathbb{R}, \vec{x} \in S \Rightarrow a\vec{x} \in S$

Remark: $\vec{0} \in S$ for all subspaces S

$a=0 \quad \vec{x} \in S \Rightarrow a \cdot \vec{x} = \vec{0} \in S$

Examples: planes, lines in $\mathbb{R}^3$

Theorem.

$A \in \mathbb{R}^{m \times n}$

$S = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$

(all solutions of the homogeneous system) is a subspace

Proof: Show that (1) + (2) hold

(1) Let $\vec{x}, \vec{y} \in S$ then $A\vec{x} = \vec{0}$, $A\vec{y} = \vec{0}$, then

$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0}$$
$$= \vec{0} \quad \therefore \vec{x} + \vec{y} \in S$$

(2) $a \in \mathbb{R}$, $\vec{x} \in S$ then $A\vec{x} = \vec{0}$.

$$A(a\vec{x}) = a\underbrace{A\vec{x}}_{\vec{0}} = a\vec{0} = \vec{0}$$

$\therefore a\vec{x} \in S$

$\to$ S is a subspace

Def.: Nullspace of a matrix $\{\vec{x} \mid A\vec{x} = \vec{0}\}$

Def.: For a linear transformation

$$T: \mathbb{R}^n \to \mathbb{R}^m$$

$$\text{Null}(T) = \{\vec{x} \in \mathbb{R} \mid T(\vec{x}) = \vec{0}\}$$

$$= \{\vec{x} \in \mathbb{R} \mid [T]\vec{x} = 0\}$$

is called the nullspace of T.

Example:

$$T(x_1, x_2) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}$$

Find Null(T), find $\vec{x}$ with $T(\vec{x}) = \vec{0}$

Solve

$x_1 + x_2 = 0$

$x_1 - x_2 = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$$

$x_2 = 0$, $x_1 = 0$

$\Rightarrow \text{Null}(T) = \{\vec{0}\}$

Let $\vec{b} \neq \vec{0}$, $\vec{b} \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$

What are the solutions of
$A\vec{x} = \vec{b}$

Theorem: $A \in \mathbb{R}^{m \times n}$, $\vec{b} \in \mathbb{R}^m \setminus \{\vec{0}\}$

Let $A\vec{p} = \vec{b}$, $\vec{p} \in \mathbb{R}^n$

(1) If $\vec{v}$ is another solution of $A\vec{x} = \vec{b}$ then $\vec{v} - \vec{p}$ is a solution of the homogeneous system. (i.e. $A(\vec{v} - \vec{p}) = \vec{0}$)

(2) If $\vec{h} \in \text{Null}(A)$, then $\vec{p} + \vec{h}$ is a solution of $A\vec{x} = \vec{b}$. (i.e. $A(\vec{p} + \vec{h}) = \vec{b}$)

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Def.: $L : \mathbb{R}^n \to \mathbb{R}^m$ then

the range of L

$$\text{Range}(L) = \{\vec{y} \in \mathbb{R}^m \mid L(\vec{x}) = \vec{y}\}$$

Example:

$L = \text{proj}_{\vec{v}}$, then

$L(\vec{x})$ is always a multiple

of $\vec{v}$ $\therefore$ $\text{Range}(\text{proj}_{\vec{v}}) =$

$\text{span}(\{\vec{v}\})$

$$\text{Range}(L) \underset{\text{subset}}{\subset} \text{codomain}(L)$$

Ex.:

$$L(x_1, x_2) = \begin{bmatrix} x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

$$[L] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$\vec{y} \in \text{Range}(L)$ if there

is an $\vec{x}$ s.t. $[L]\vec{x} = \vec{y}$

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{y}$$

i.e. if $\vec{y} \in \text{span}$ (column vectors of $[L]$)

$$\left.\begin{matrix} 1 & +1 \\ 1 & -1 \end{matrix}\right| \begin{matrix} y_1 \\ y_2 \end{matrix} \longrightarrow \left.\begin{matrix} 1 & 1 \\ 0 & -2 \end{matrix}\right| \begin{matrix} y_1 \\ y_2 \end{matrix}$$

This has a solution for all

$y_1, y_2 \longrightarrow \text{Range}(L) = \mathbb{R}^2$

Example:

$$L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad [L] = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

$$\left.\begin{matrix} 1 & -3 \\ -2 & 6 \end{matrix}\right| \begin{matrix} y_1 \\ y_2 \end{matrix} \longrightarrow \left.\begin{matrix} 1 & -3 \\ 0 & 0 \end{matrix}\right| \begin{matrix} y_1 \\ y_2 + 2y_1 \end{matrix}$$

This system is consisten, only if $y_2 + 2y_1 = 0$ i.e. $y_2 = -2y_1$

let $y_1 = t$, $t \in \mathbb{R}$

$$\vec{y} = t \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ t \in \mathbb{R}$$

$$\text{Range}(L) = \text{span}\left(\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right\}\right)$$

$$= \text{span}\left(\left\{\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}\right\}\right)$$

$A \in \mathbb{R}^{m \times n}$, then the non-zero rows of the REF of A form a basis of row(A).

$\therefore$ $\dim(\text{row}(A)) = \text{rank}(A)$

$= \#$ of leading entries in REF

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ -1 & 1 & -4 \end{pmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} \right\}$$

is a basis of row(A).

Theorem: $A \in \mathbb{R}^{m \times n}$ with REF B.

The columns of A correspon-ding to the columns in B with leading entries form a basis of $\text{col}(A)$.

$\rightarrow$ Continue Ex.:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ -1 & 1 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rightarrow$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis of $\text{col}(B)$.

Comment: $\dim(\text{col}(A)) = \text{rank}(A) = \dim(\text{row}(A))$

Remark:

$$\text{col}(A) = \text{row}(A^T)$$
$$\text{row}(A) = \text{col}(A^T)$$

Observation:

$n - \text{rank}(A) = \#$ of parameters in general solution of $A\vec{x} = \vec{0}$

$= \#$ of vectors spanning $\text{Null}(A)$

$\overset{?}{=} \dim(\text{Null}(A))$

Theorem: $A \in \mathbb{R}^{m\times n}$ with $\text{rank}(A) = r$

then $\dim(\text{Null}(A)) = n - r$

- The vectors in the general solution of $A\vec{x} = \vec{0}$ form a basis of $\text{Null}(A)$

Rank Theorem:

$$\text{rank}(A) + \dim(\text{Null}(A)) = n$$

Example:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\text{rank}(A) = 3 = \dim(\text{row}(A))$$

$$= \dim(\text{col}(A))$$

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \right\}$$

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \right\}$$

$$\dim(\text{Null}(A)) = 4 - 3 = 1$$

$$x_4 = 0, \quad x_3 = t, \quad x_2 = 4t, \quad x_1 = 7t$$

$$\text{Null}(A) = \text{span}\left\{ \begin{bmatrix} 7 \\ 4 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Def.:
The columnspace of a matrix $A \in \mathbb{R}^{m \times n}$, col(A), is the span of the columns of A.

Theorem: Let $L: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, then

$$\text{range}(L) = \text{col}([L])$$

Def.: The rowspace of a matrix $A \in \mathbb{R}^{m \times n}$, row(A), is the space spanned by the rows of A.

Example:

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$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^2$$

$$\text{row}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\} \subset \mathbb{R}^3$$

- $\text{row}(A) = \text{col}(A^T)$

Theorem: Elementary row operations do not change the row space of a matrix.

Bases, dimension

Def.:
- Let S be a subspace, then $B = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis of S if B is linear independent, and $S = \text{span}(B)$.
- $\dim(S)$ = # of vectors in a basis

⌈Remark: The definition is only meaningful if each basis of S has the same number of vectors, which is true, but we will not prove⌋