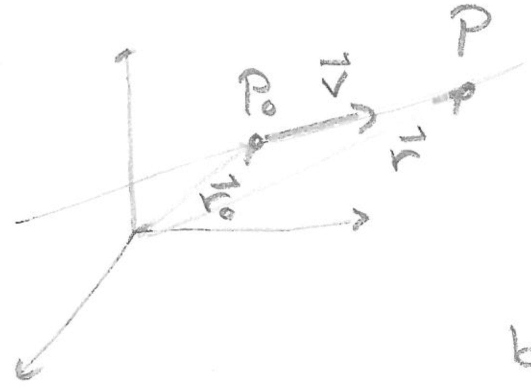
Equation of lines (in $\mathbb{R}^3$)



$P_0(x_0, y_0, z_0)$

A line is determined by a point P_0 (fixed point) on the line and a vector, $\vec{v}$, indicating the direction.

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Let $P(x, y, z)$ be a point on L, with location vector $\vec{r}$, then

$$\vec{r} = \vec{r_0} + t \cdot \vec{v}, \text{ for some number } t.$$

We can permit any value in $\mathbb{R}$ for t to obtain all points on L.

t is a parameter in this equation (vector parametric eq.)

Use $\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\vec{r}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$ and

$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in the parametric eq.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \iff$$

$$\left.\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}\right\}$$ <u>scalar parametric eq.</u>

Assume $a, b, c \neq 0$, then

$$t = \underbrace{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

"symmetric equations for a line"

<u>Example</u>: The line with point $P_0(1,2,3)$ in direction of $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ can be given as

$$\vec{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$ "vector parametric"

or,

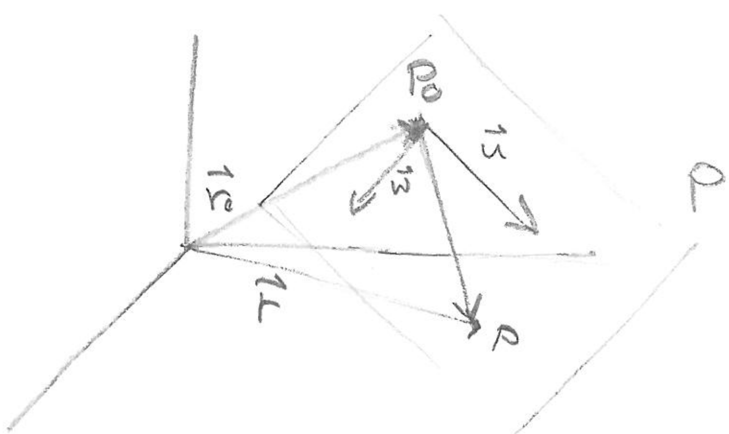
$x = 1 + t, \quad y = 2 - t, \quad z = 3 + 2t$

"scalar parametric"

or

$$\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z-3}{2}$$ "symmetric equs"

Equation of planes in $\mathbb{R}^3$



The plane P is determined by a point P_0 and two vectors, $\vec{u}$ and $\vec{v}$, parallel to P.

$$\vec{u} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad P_0(x_0, y_0, z_0),$$

with $\vec{r_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \overrightarrow{OP_0}$

The points in the plane, P, are then all points $P(x, y, z)$ with location vector, $\vec{r} = \overrightarrow{OP}$

$$\vec{r} = \vec{r_0} + s\vec{u} + t\vec{v}, \quad \underbrace{s, t}_{\text{parameters}} \in \mathbb{R}$$

"vector parametric eq."

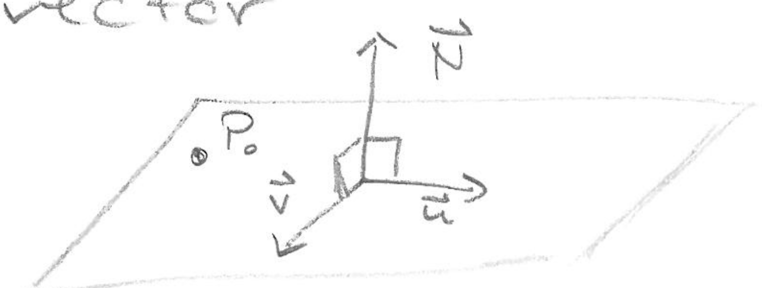
Putting in the coordinates

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + s \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + t \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

or

$$\left.\begin{aligned} x &= x_0 + s a_1 + t b_1 \\ y &= y_0 + s a_2 + t b_2 \\ z &= z_0 + s a_3 + t b_3 \end{aligned}\right\} \text{"scalar parametric eqns"}$$

An alternative method of describing a plane is by its normal vector



$$\vec{N} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

then $\vec{V} = \overrightarrow{OB} - \overrightarrow{OA}$ is in the plane

and $\vec{N} = \vec{u} \times \vec{v}$ is a normal vector

for the plane.

$$\vec{V} = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 0 \\ -1 & 2 & 2 \end{vmatrix} = \begin{vmatrix} -2 & 0 \\ 2 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ -1 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & -2 \\ -1 & 2 \end{vmatrix} \vec{k}$$

$$= \underset{A}{(-4)}\vec{i} + \underset{B}{2}\vec{j} + \underset{C}{(-4)}\vec{k}$$

Normal eq (use $P_0 = A$)

$$-4(x-1) + 2(y-0) - 4(z-2) = 0$$

simplifies to

$$-4x + 4 + 2y - 4z + 8 = 0$$

$$\boxed{-4x + 2y - 4z + 12 = 0}$$

get a normal vector and a point C

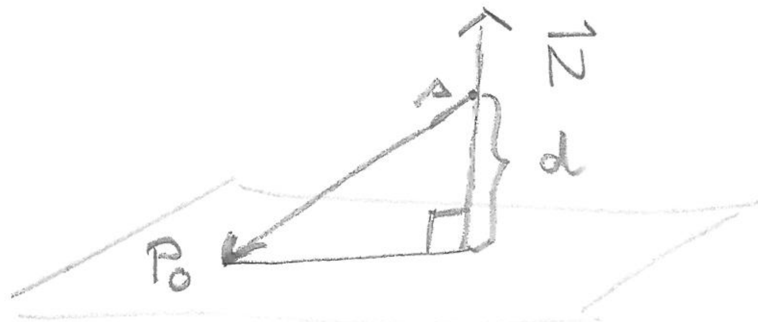
from the normal eq. :

$$\vec{N} = \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ -4 \end{pmatrix} \quad \text{for } \overrightarrow{OC} = \begin{pmatrix} 0 \\ 0 \\ c_3 \end{pmatrix} \Big\} \text{ I choose}$$

then $0 + 0 - 4c_3 + 12 = 0 \rightarrow c_3 = 3$

$C = (0, 0, 3)$ is on the plane.

Distance of a point A to a plane P



$$d = \left| \text{proj}_{\vec{N}} \overrightarrow{AP_0} \right|$$

continue example with $A(3,3,3)$
$P_0(0,0,3)$

$$\overrightarrow{AP_0} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -3 \\ 0 \end{bmatrix} \qquad \vec{N} = \begin{bmatrix} -4 \\ 2 \\ -4 \end{bmatrix}$$

$$\text{proj}_{\vec{N}} \overrightarrow{AP_0} = \frac{\vec{N} \cdot \overrightarrow{AP_0}}{\| \vec{N} \|} = \frac{12 + (-6) + 0}{\sqrt{16 + 4 + 16}}$$

$$= \frac{6}{\sqrt{36}} = 1$$

$\Rightarrow d = 1$