Inverse Linear Transf.

Def.: $Id : \mathbb{R}^n \to \mathbb{R}^n$ is called the Identity (transformation) if $Id(\vec{x}) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ (gives the identical vector)

$[Id] = I_n$

Def.: If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a lin. transf. and there exists $M : \mathbb{R}^m \to \mathbb{R}^n$, s.t. $L \circ M = Id$, then L is <u>invertible</u>, and M is called its <u>inverse</u>.

This means, if

$L(\vec{v}) = \vec{w}$, then $M(\vec{w}) = \vec{v}$

Theorem:

Suppose $L: \mathbb{R}^n \to \mathbb{R}^n$ linear transf. with standard matrix $[L]$, and $M: \mathbb{R}^n \to \mathbb{R}^n$ with standard matrix $[M]$, then M is an inverse of L if and only if $[M]$ is the inverse matrix of $[L]$.

Example:

$L: \mathbb{R}^2 \to \mathbb{R}^2$ Stretch by k

$\Rightarrow [L] = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ (why?)

then M, stretch by $\frac{1}{k}$ is the inverse $[M] = \begin{bmatrix} 1/k & 0 \\ 0 & 1/k \end{bmatrix}$

with $[M][L] = I_2$.

Example:

$L: \mathbb{R}^2 \to \mathbb{R}^2$ reflection

on $x_1 = x_2$, $L(\vec{e}_1) = \vec{e}_2$, $L(\vec{e}_2) = \vec{e}_1$

$[L] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\left.\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\;\right|\;\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \;\; \underset{R}{\overset{R_1}{\updownarrow}} \;\; \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\;\left|\;\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right.$$

$\Rightarrow [M] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [L] \Rightarrow$

M is reflection on $x_1 = x_2$.

Theorem (Invertible Matrix Theorem, cont.)

Suppose $L: \mathbb{R}^n \to \mathbb{R}$ lin. transf. with $[L] = A$ then the following statements are equivalent

(1) A is invertible

(8) L is invertible

(9) $\text{Range}(L) = \mathbb{R}^n$

(10) $\text{Null}(L) = \{\vec{0}\}$

Proof: (1) equiv (7) (theorem above)

(1) $\Leftrightarrow$ (9) because $A\vec{x} = \vec{b}$ has a solution for all $\vec{b}$, if and only if A invertible.

(9) $\Leftrightarrow$ (10) Rank theorem.

Ex.: $L: \mathbb{R}^2 \to \mathbb{R}^2$ $\vec{v} \in \mathbb{R}^2$

$L(\vec{x}) = \text{proj}_{\vec{v}}(\vec{x})$, then

L is not invertible.

For all $\vec{x}$ $L(\vec{x}) = t \cdot \vec{v}$, for some t. $\Rightarrow$ If $\vec{y} \neq t\vec{v}$

$\vec{y} \notin \text{Range}(L) \Rightarrow$

$\text{Range}(L) \neq \mathbb{R}^2$

$\Rightarrow$ L not invertible

Ex.: $L: \mathbb{R}^2 \to \mathbb{R}^2$ with $[L] = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$

is invertible, because

$[L]\vec{x} = \vec{0}$ has exactly one solution, $\vec{x} = \vec{0}$ $\Rightarrow$ $\text{Null}(\vec{L}) = \{\vec{0}\}$

$\Rightarrow$ L is invertible.