Friday

Inverse matrices

When $a, b \in \mathbb{R}$ $a \neq 0$ and

you solve $ax = b$, then you

find $x = \frac{b}{a} = a^{-1}b$

$a \neq 0$ is important.

What about

$$A\vec{x} = \vec{b} \ ?$$

Can we use a similar approach?

What does "$A \neq 0$" translate to?

Definition $A \in \mathbb{R}^{n \times n}$.

If there exists a matrix $B \in \mathbb{R}^{n \times n}$ with

$AB = BA = I_n$, then

A is called invertible, and

$A^{-1} = B$ is called its inverse.

Example:

$$A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$$

then

$$AA^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \text{, also}$$

$$A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A is invertible with inverse A^{-1},

Theorem (the inverse is unique)
Let $A \in \mathbb{R}^{n \times n}$ with $B, C \in \mathbb{R}^{n \times n}$

$$AB = BA = I = AC = CA$$

then $B = C$

Proof: $B = BI = B(AC) = (BA)C$

$$= IC = C \text{ so } B = C.$$

Theorem:

$A, B \in \mathbb{R}^{n\times n}$ s.t. $AB = I$

then $BA = I$.

(*therefor* $B = A^{-1}$),

and $\mathrm{rank}(A) = \mathrm{rank}(B) = n$.

Theorem: $A, B \in \mathbb{R}^{n\times n}$ invertible
$t \in \mathbb{R}, t \neq 0$

(1) $(tA)^{-1} = \frac{1}{t}A^{-1}$

(2) $(AB)^{-1} = B^{-1}A^{-1}$

(3) $(A^T)^{-1} = (A^{-1})^T$

Proof: (2) only

$$(AB)(B^{-1}A^{-1}) = A\overbrace{BB^{-1}}^{I}A^{-1}$$

$$= AIA^{-1} = AA^{-1} = I$$

$$\Rightarrow B^{-1}A^{-1} = (AB)^{-1}$$

Find A^{-1}

If $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = e_2$ $(A \in \mathbb{R}^{2\times 2})$

then $A\left[\vec{x}_1 \mid \vec{x}_2\right] = \left[\vec{e}_1 \mid \vec{e}_2\right] = I_2$

To find the inverse solve

$$A\vec{x}_1 = \vec{e}_1 , \quad A\vec{x}_2 = \vec{e}_2$$

simultaneously.

Algorithm:

(1) Row reduce $\left[A \mid I_n\right]$ so that the left is in Reduced Row Echelon Form

(2) If the RREF is of the form $\left[I_n \mid B\right]$, then $A^{-1} = B$

(3) If the left side is not I_n, A is not invertible.

Example: $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, rank = 2

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 0 & \frac{1}{2} \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ rank = 1

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

$\to$ not invertible.

__Remark__: $A \in \mathbb{R}^{n \times n}$ invertible, then $\vec{x} = A^{-1}\vec{b}$ is a solution of $A\vec{x} = \vec{b}$, because

$$A\vec{x} = A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I\vec{b} = \vec{b}.$$

Theorem: (Invertible Matrix Th.) 6

$A \in \mathbb{R}^{n \times n}$, then the following statements are equivalent

(1) A is invertible

(2) $\text{rank}(A) = n$

(3) RREF of $A = I_n$

(4) $A\vec{x} = \vec{b}$ is consisten for all $\vec{b}$ and has a unique solution.

(5) The columns are lin. ind.

(6) $\text{Col}(A) = \mathbb{R}^n$

(7) the columns of A form a basis of $\mathbb{R}^n$.

Continue example:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$, find $\vec{x}$

with $A\vec{x} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

$$A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \vec{x} = A^{-1} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(test

$$A\vec{x} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \checkmark)$$