

1 Introduction

Math 120 – Basic Linear Algebra I

- This is a math class, expect abstract thinking and deductive reasoning – a lot!!
- You will learn to prove theorems, and to apply their results in solving problems.
- Some examples, but focus on the theory
- no calculator
- Read the outline
- contact information, use email
- webpage, class notes to be posted through the term + homework solutions.
- dates for exams(2 midterm + 1 final), and assignments

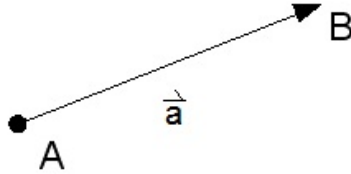
2 Geometric Vectors

2.1 Introduction

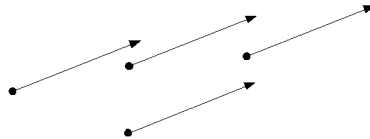
Geometric vectors are defined as line segments with a direction (or arrows).

A geometric vector is characterized by two properties, its magnitude (norm, $||\cdot||$) and its direction. The length of the vector describes its magnitude and the direction of the arrow defines the vector's direction.

Use lowercase letters with an arrow on top to represent vectors.



In the figure above A is the initial point and B is the terminal point of the vector $\vec{a} = \overrightarrow{AB}$. Vectors of the same length and direction are called equivalent.



For equivalent vectors \vec{v} and \vec{w} we write $\vec{v} = \vec{w}$.

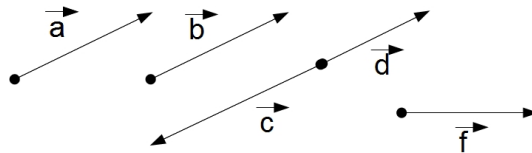
Definition 1

A vector whose initial and end point are the same is called a zero vector, $\vec{0} = \overrightarrow{AA}$.

The norm of the zero vector is 0, write $||\vec{0}|| = 0$, the direction of the zero vector is not defined.

Definition 2

Two vectors are collinear, if they lie on the same line or parallel lines.



In the figure above all vectors but \vec{f} are collinear to each other.

Definition 3

Two collinear vectors are called co-directed if they have the same direction. They are oppositely directed otherwise.

For example in the diagram above vectors \vec{a} , \vec{b} , and \vec{d} are co-directed, whereas those three vectors are oppositely directed to vector \vec{c} .

We write \vec{a} upuparrows \vec{b} and \vec{a} updownarrows \vec{c} .

Remark: co-directed (parallel) vectors are colinear AND point in the same direction
oppositely directed arrows are colinear AND point in different directions.

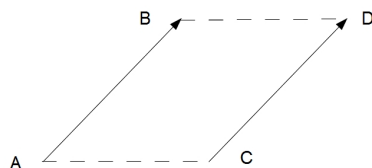
Definition 4

Two vectors are equal, if they are parallel and of equal length.

(we called them equivalent above)

Therefore, if we translate (move) a vector it is still considered the same vector, (the location does not matter only direction and length).

Remark: Given four mutually different points A, B, C, D , where $\overrightarrow{AB} = \overrightarrow{CD}$,
Then the figure $ABDC$ is a parallelogram, because \overrightarrow{AB} and \overrightarrow{CD} are parallel and of equal length.

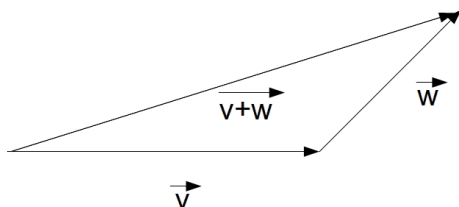


In the case the two vectors are not equal the figure $ABDC$ is NOT a parallelogram. Why? Therefore we have another characterization when two vectors are equal.

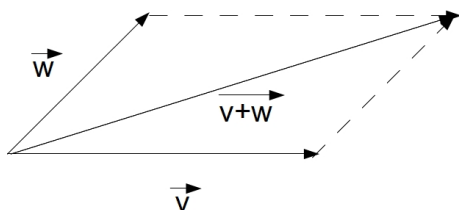
2.2 Addition of Vectors

Definition 5

If \vec{v} and \vec{w} are any vectors then the sum of the vectors, $\vec{v} + \vec{w}$, is the vector that has the same initial point as \vec{v} and the same final point as \vec{w} , when the initial point of \vec{w} coincides with the final point of \vec{v} .



The parallelogram approach gives you the same result:



Theorem 1

1. Commutative law of addition.

For any vectors \vec{v} and \vec{w}

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

2. Associative law of addition.

For any vectors \vec{v}, \vec{w} and \vec{x}

$$\vec{v} + (\vec{w} + \vec{x}) = (\vec{v} + \vec{w}) + \vec{x}$$

3. For any vector \vec{v}

$$\vec{v} + \vec{0} = \vec{v}$$

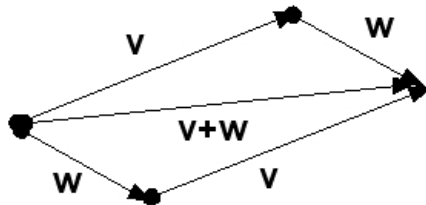
4. Existence of the inverse.

For every vector \vec{v} exists a vector $-\vec{v}$, so that

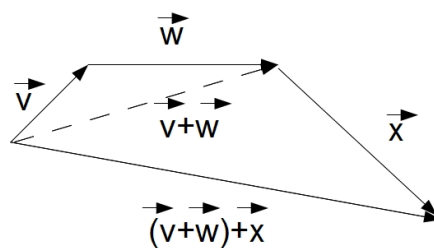
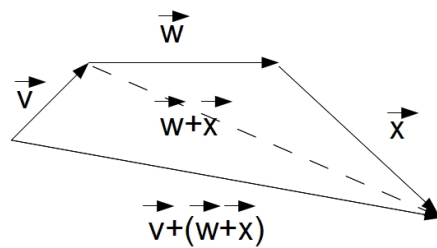
$$\vec{v} + (-\vec{v}) = \vec{0}$$

"Proof":

- 1.



- 2.



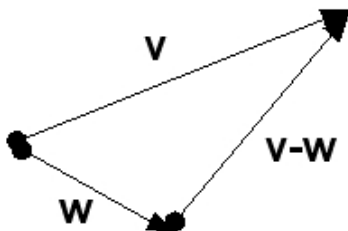
3. Let $\vec{v} = \overrightarrow{AB}$ and $\vec{0} = \overrightarrow{BB}$, then $\vec{v} + \vec{0} = \overrightarrow{AB} = \vec{v}$
4. Let $\vec{v} = \overrightarrow{AB}$, set $-\vec{v} = \overrightarrow{BA}$, then $\vec{v} + (-\vec{v}) = \overrightarrow{AA} = \vec{0}$.

Definition 6

For any vectors \vec{v} and \vec{w} , the difference is

$$\vec{v} - \vec{w} = \vec{v} + (-\vec{w})$$

To obtain $\vec{v} - \vec{w}$ from \vec{v} and \vec{w} without $-\vec{w}$, then the vector with initial point at terminal point of \vec{w} and terminal point being the terminal point of \vec{v} is $\vec{v} - \vec{w}$



Example 1

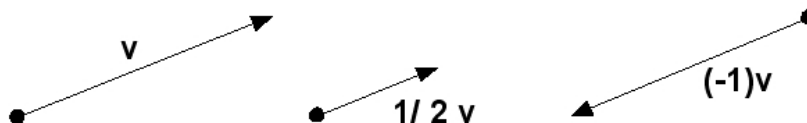
Complex example.

2.3 Scalar Multiplication

Definition 7

If \vec{v} is a nonzero vector and $k \in \mathbb{R} \setminus \{0\}$, then the product $k\vec{v}$ is defined to be the vector that has $|k|$ times the length of \vec{v} and the same direction as \vec{v} , if $k > 0$ and the opposite direction if $k < 0$. If $k = 0$ or $\vec{v} = \vec{0}$, then define $k\vec{v} = \vec{0}$, the zero vector.

Write $\|k\vec{v}\| = |k| \times \|\vec{v}\|$, if $|k| > 0$ then $k\vec{v}$ uparrow \vec{v} , and if $k < 0$ then $k\vec{v}$ updownarrow \vec{v} .



Theorem 2

(Without proof) Two vectors are proportional (i.e. one of them is a scalar multiple of the other) if and only if they are collinear.

Think about the different definitions of the two terms used in this theorem and see how they relate.

Theorem 3

Properties of scalar multiplication.

5. For any vector \vec{v} , $1 \cdot \vec{v} = \vec{v}$.
6. (Associative law) For any vector \vec{v} and numbers $k, l \in \mathbb{R}$, $(kl)\vec{v} = k(l\vec{v})$.
7. (Distributive law) For any vector \vec{v} and numbers $k, l \in \mathbb{R}$, $(k + l)\vec{v} = k\vec{v} + l\vec{v}$.
8. (Distributive law) For any vectors \vec{v}, \vec{w} and number $k \in \mathbb{R}$, $k(\vec{v} + \vec{w}) = k\vec{v} + k\vec{w}$.

Proof (Just 2 examples, to give you an idea):

5. $1 \cdot \vec{v}$ is by definition the vector of length $|1| \cdot \|\vec{v}\| = \|\vec{v}\|$, and the two vectors are by definition of the scalar product are collinear and since $1 > 0$ it has the same direction of \vec{v} , therefore a parallel vector of equal length, which by definition is equal to \vec{v} .
6. We need to show that the two vectors have the same length and that they are parallel.

Length: (Think about each step!)

$\|(kl)\vec{v}\| = |kl| \cdot \|\vec{v}\| = |k| \cdot |l| \cdot \|\vec{v}\| = |k|(|l| \cdot \|\vec{v}\|) = |k| \cdot \|l\vec{v}\| = \|k(l\vec{v})\|$. Comparing the first and last expression in the equation is telling that the length of the two vectors are the same.

Parallel: By definition $(kl)\vec{v}$ is collinear to \vec{v} , which is collinear to $l\vec{v}$, which is collinear to $k(l\vec{v})$, therefore the two vectors, $(kl)\vec{v}$ and $k(l\vec{v})$, are collinear.

But are they parallel or anti-parallel? (We need parallel). Here we need to consider four cases.

(i) $k, l > 0$, (ii) $k, l < 0$ (iii) $k > 0, l < 0$, and (iv) $k < 0, l > 0$

Only do (i) here. Because $k > 0, l > 0$, $kl > 0$, so $(kl)\vec{v}$ upup \vec{v} , and $l\vec{v}$ upup \vec{v} , and $k(l\vec{v})$ upup $l\vec{v}$, together

$(kl)\vec{v}$ upup \vec{v} upup $l\vec{v}$ upup $k(l\vec{v})$.

The other proves try as exercise.

Example!! $(-1)\vec{v} = -\vec{v}$

The seven properties of addition and scalar multiplication are the defining properties of vectors. Later in this course we will look at very different objects with the same properties and call them vectors as well.

2.4 Linear Combinations

Definition 8

Let $\vec{e}_1, \dots, \vec{e}_n$ be vectors, and $c_1, \dots, c_n \in \mathbb{R}$, then

$$\vec{a} = c_1\vec{e}_1 + \dots + c_n\vec{e}_n$$

is called a linear combination of $\vec{e}_1, \dots, \vec{e}_n$ with coefficients c_1, \dots, c_n .

Examples:

1. $k\vec{e}$

2. $\vec{u} + \vec{v}$
3. $\vec{u} - \vec{v}$
4. $2\vec{u} + (-1/2)\vec{v}$

Theorem 4

Given a vector \vec{e} , then any vector \vec{a} collinear to \vec{e} can be written as $\vec{a} = c\vec{e}$, for a unique $c \in \mathbb{R}$.

This means that if we have two collinear vectors, then they are linear combinations of each other.

It follows:

- For two collinear vectors \vec{e} and \vec{a} : $\vec{a} = c\vec{e}$, then $\vec{a} - c\vec{e} = \vec{0}$.

This is saying that you can always find a linear combination of collinear vectors \vec{e}, \vec{a} that combine to $\vec{0}$. (coefficients $c_1 = 1, c_2 = -c$, observe $c_1 \neq 0$).

- What happens if we have two vectors \vec{e}, \vec{a} , which are **not** collinear, can we find a linear combination, so the result is $\vec{0}$?

Yes, $0\vec{e} + 0\vec{a} = \vec{0}$ (both coefficients are 0). Could there be other coefficients (at least one is not zero)?

Let's try:

$$c_1\vec{e} + c_2\vec{a} = \vec{0}, c_1 \neq 0$$

Add $-c_2\vec{a}$ on both sides:

$$\begin{aligned} c_1\vec{e} + \underbrace{c_2\vec{a} - c_2\vec{a}}_{=\vec{0}} &= \underbrace{\vec{0} - c_2\vec{a}}_{-c_2\vec{a}} \\ \Leftrightarrow c_1\vec{e} &= -c_2\vec{a} \end{aligned}$$

Divide both sides by c_1 , which is possible because we assumed that it is not 0

$$\Leftrightarrow \vec{e} = -(c_2/c_1)\vec{a}$$

Therefore \vec{a} is a linear combination of \vec{e} , therefore the two vectors are collinear (see theorem above), but we assumed they are not collinear.

In summary: If we assume that $c_1 \neq 0$, we end in a contradiction, which tells us, that the assumption can not be right, so c_1 must be zero.

Result:

If we have two vectors which are not collinear then the only way you can combine them linearly to make $\vec{0}$ is to use coefficients of 0.

In summary:

We can find coefficients different from zero to combine two collinear vectors to $\vec{0}$, but this is not possible for non collinear vectors.

We found a different way to characterize collinear vectors!

Definition 9

A set of vectors $\{\vec{e}_1, \dots, \vec{e}_n\}$ is called linearly independent, if from

$$c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n = \vec{0}$$

follows that $c_1 = c_2 = \dots = c_n = 0$.

Otherwise the set is called linearly dependent.

Example 2

1. Collinear(linearly dependent) versus not collinear(linearly independent)
2. $\vec{w} = 2\vec{u} + 3\vec{v}$, then $\{\vec{w}, \vec{u}, \vec{v}\}$ is linearly dependent. Why?

The following statements are always true:

1. Any set of vectors including the zero vector is linearly dependent.
2. Any set of vectors that contains a linearly dependent subset is linearly dependent.
3. A set of vectors is linearly dependent if and only if one of them is a linear combination of the others.

Proof:....

Example 3

- A linearly independent set of one vector is one non-zero vector.
- A linearly independent set of two vectors is a set of non-collinear vectors.
- Sets of linearly dependent and independent vectors.

Definition 10

The set of all linear combinations of a given set of vectors is called the span of these vectors.

If \vec{v} is a linear combination of vectors $\{\vec{e}_1, \dots, \vec{e}_n\}$, write $\vec{v} \in \text{Span}(\{\vec{e}_1, \dots, \vec{e}_n\})$.

We say that $\{\vec{e}_1, \dots, \vec{e}_n\}$ span the $\text{Span}(\{\vec{e}_1, \dots, \vec{e}_n\})$.

Example 4

1. one vector
2. two vectors
3. three vectors

Theorem 5

Let \vec{e} be a non-zero vector, then the $\text{Span}(\{\vec{e}\})$ is the set of all vectors collinear to \vec{e} , which is the line collinear to the vector (line of vectors).

Theorem 6

Let \vec{e}_1, \vec{e}_2 be non collinear, then $\text{span}(\{\vec{e}_1, \vec{e}_2\})$ is the plane the two vectors fall within.

2.5 The concept of a Basis

Here we combine the two concepts of linear independence and spanning.

Definition 11

A basis for a line, plane, etc. is a set of linearly independent vectors spanning the line, plane, etc.

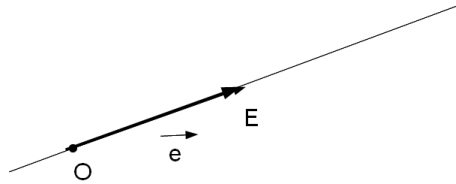
Remark: If \mathcal{B} is a basis of a line, plane, etc. then every vector \vec{a} in the line, plane, etc. can be written as

$$\vec{a} = c_1\vec{e}_1 + \dots + c_n\vec{e}_n$$

(c_1, \dots, c_n) are called the coordinates of \vec{a} w.r.t. \mathcal{B} .

Example 5

1. According to the theorem above the basis of a line consists of a non-zero vector collinear to the line.



All vectors on the line can be translated so that their initial point coincides with the origin, O , which is the initial point of the vector in the basis. Call this a position vector.

The line becomes the coordinate axis, with origin O , and each vector on the line is characterized by the coordinate of the position vector w.r.t basis $\mathcal{B} = \{\vec{e}\}$.

If $\vec{a} = c\vec{e}$, then \vec{a} is characterized by the coordinate: $\vec{a} \leftrightarrow c$.

Using the coordinate axis:

Also, if $\vec{a} = a\vec{e}$ and $\vec{b} = b\vec{e}$, then $\vec{a} + \vec{b} = (a + b)\vec{e}$: $\vec{a} + \vec{b} \leftrightarrow (a + b)$.

And, if $\vec{a} = a\vec{e}$, then $k\vec{a} = (ka)\vec{e}$: $k\vec{a} \leftrightarrow (ka)$.

2. plane of vectors

The basis of a plane consists of two vectors non-collinear from that plane.

Theorem 7

Every set of two noncollinear vectors form a basis of the plane they fall within.

Which means that every vector which is coplanar with the two vectors can be expressed as a linear combination of the two.

Remark: Any three coplanar vectors are linearly dependent.

In conclusion a basis of a plane consists of two noncollinear vectors.

Similar as for the line, we can now say that the two vectors of a plane define a **coordinate system**.

Assume the basis is $\{\vec{e}_1, \vec{e}_2\}$, then every vector \vec{a} in the plane can be expressed as a linear combination in \vec{e}_1 and \vec{e}_2 .

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$$

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ are the the coordinates of \vec{a} in the coordinate system induced by $\{\vec{e}_1, \vec{e}_2\}$. They are the coordinates w.r.t. $\{\vec{e}_1, \vec{e}_2\}$.

$$\vec{a} \leftrightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

It is also true that Also, if $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$ and $\vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2$, then $\vec{a} + \vec{b} = (a_1 + b_1)\vec{e}_1 + (a_2 + b_2)\vec{e}_2$:

$$\vec{a} + \vec{b} \leftrightarrow \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix}$$

And, if $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$, then $k\vec{a} = (ka_1)\vec{e}_1 + (ka_2)\vec{e}_2$:

$$k\vec{a} \leftrightarrow \begin{bmatrix} ka_1 \\ ka_2 \end{bmatrix}$$

And in general:

$$c_1\vec{a} + c_2\vec{b} \leftrightarrow \begin{bmatrix} c_1a_1 + c_2b_1 \\ c_1a_2 + c_2b_2 \end{bmatrix}$$

Because of this one-to-one relationship it is sufficient to to give the basis and the coordinates of the vector w.r.t. this basis to identify a vector.

Definition 12

The standard basis in \mathbb{R}^2 is the set of vectors $\{\vec{i}, \vec{j}\}$ with

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$