Determinants

We had • $A \in \mathbb{R}^{2\times 2}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{then}$$

$$\det A = a_{11}a_{22} - a_{21}a_{12}$$

• $A \in \mathbb{R}^{3\times 3}$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{then}$$

$$\det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Def.: Let $A \in \mathbb{R}^{n\times n}$

$A(i,j)$ is the submatrix of A deleting row i and column j.

Define the cofactor

$$C_{ij} = (-1)^{i+j} \det A(i,j)$$

Then $A \in \mathbb{R}^{3\times 3}$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

(expansion along the first row)

Example:

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 3 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$C_{11} = (-1)^{1+1} \cdot 7 = 7$$

$$C_{12} = (-1)^{1+2} \cdot 8 = -8$$

$$C_{13} = (-1)^{1+3}\, 4 = 4$$

$$\Rightarrow \det A = 2 \cdot 7 + 1 \cdot (-8) + 5(4)$$

$$= 14 - 8 + 20 = 26$$

Def.: The determinant of $A \in \mathbb{R}^{n\times n}$ is defined by

$$\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

Example:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & -1 & 2 \\ -1 & -2 & 0 & 3 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$= 0 \cdot C_{11} + 0\, C_{12} + 1 \cdot C_{13} + 0 \cdot C_{14}$$

$$= \underbrace{1 \cdot \underset{=1}{(-1)^{1+3}}}_{=1} \cdot \begin{vmatrix} 2 & 0 & 2 \\ -1 & -2 & 3 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 2\,(-1)^{1+1} \begin{vmatrix} -2 & 3 \\ 0 & 1 \end{vmatrix} + 0 \;\ldots\; + 2\,\overset{(-1)^{1+3}}{\begin{vmatrix} -1 & -2 \\ 0 & 0 \end{vmatrix}}$$

$$= 2(-2) + 2(0) = -4$$

Theorem (we can expand any row or column) $A \in \mathbb{R}^{n \times n}$

expansion along i-th row

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

expansion along j-th column

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

Cont. EX.:

expansion along 2^{nd} column

$$\det A = 0\,C_{12} + 0\,C_{22} + (-2)\,C_{32} + 0\,C_{42}$$

$$= \underbrace{(-2)\,(-1)^{3+2}}\det\begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= 2 \cdot \quad 1 \cdot (-1)^{1+2} \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}$$

expand along row 1

$$= 2\,(-1)\,2 = \underline{\underline{-4}}\,.$$

Theorem: $A \in \mathbb{R}^{n\times n}$

If one row (column) contains only zeros then $\det(A) = 0$.

Theorem: $A \in \mathbb{R}^{n\times n}$, if A is upper or lower triangular then

$$\det(A) = a_{11}\,a_{22} \cdots a_{nn}$$

Theorem: $A \in \mathbb{R}^{n\times n}$, then

$$\det(A) = \det A^T$$

Elementary Row Operations

Theorem:

$A \in \mathbb{R}^{n\times n}$, B is obtained from by multiplying the i^{th} row by $k \in \mathbb{R}$

then $\det B = k \det A$.

("factor out k")

$$\det \begin{bmatrix} 4 & 8 \\ 3 & 4 \end{bmatrix} = 4 \det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Theorem; $A \in \mathbb{R}^{n\times n}$

$$\det(kA) = k^n \det(A)$$

Theorem: $A \in \mathbb{R}^{n\times n}$, B is obtained from by swapping two rows,

then $\det(B) = -\det(A)$

Theorem. $A \in \mathbb{R}^{n \times n}$, with two rows being equal, then $\det(A) = 0$.

Proof.: B is the matrix with the two row being swapped, the $A = B$, and

$$\det(A) = \det(B) \underset{\text{th.}}{=} -\det(A)$$

$$\Rightarrow 2\det(A) = 0 \Rightarrow \det A = 0.$$

Theorem: $A \in \mathbb{R}^{n \times n}$, B is obtained from A by adding rR_i to R_j.

Then $\det(B) = \det(A)$.

Example:

$$\det\begin{pmatrix} 1 & 2 & 4 \\ 3 & 6 & 0 \\ -1 & 3 & 2 \end{pmatrix} = 3\det\begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 0 \\ -1 & 3 & 2 \end{pmatrix}$$

$$= 3\det\begin{pmatrix} 1 & 2 & 4 \\ 0 & 0 & -4 \\ 0 & 5 & 9 \end{pmatrix} = (-3)\det\begin{pmatrix} 1 & 2 & 4 \\ 0 & 5 & 9 \\ 0 & 0 & -4 \end{pmatrix}$$

$$= (-3)\cdot 1 \cdot 5 \cdot (-4) = 60.$$

Theorem. $A \in \mathbb{R}^{n\times n}$, then the following are equivalent

(1) A is invertible

(2) $\det A \neq 0$

Theorem. $A, B \in \mathbb{R}^{n\times n}$

$$\det(AB) = \det(A)\det(B)$$

Matrix Inversion by Cofactors

Def: $A \in \mathbb{R}^{n \times n}$, then $\operatorname{cof}(A)$ is the cofactor matrix of, with

$$(\operatorname{cof}(A))_{ij} = C_{ij}$$

Ex.:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad C_{11} = 4$$

$$C_{12} = -3$$

$$\operatorname{cof}(A) = \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} \qquad C_{21} = -2$$

$$C_{22} = 1$$

Theorem: $A \in \mathbb{R}^{n \times n}$ invertible, then $A^{-1} = \frac{1}{\det(A)} (\operatorname{cof}(A))^T$

Cont Ex.: $\det(A) = -2$

$$\frac{1}{-2} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = I_2$$

Cramer's Rule

Theorem: $A \in \mathbb{R}^{n \times n}$ invertible

N_i is obtained from A by re-placing the i^{th} column by $\vec{b}$.

The $\vec{x}$ with

$$x_i = \frac{\det N_i}{\det A}$$ is a

solution of $A\vec{x} = \vec{b}$.

Example:

$$A\vec{x} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$\det A = -2$

$$N_1 = \begin{bmatrix} 2 & 2 \\ -2 & 4 \end{bmatrix} \quad \det N_1 = 12$$

$$N_2 = \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} \quad \det N_2 = -8$$

$$\Rightarrow \quad \vec{x} = \begin{pmatrix} 12/-2 \\ -8/-2 \end{pmatrix} = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$ is the solution

Determinant - Volume & Area

Earlier:

$\mathbb{R}^2$ $\vec{u}, \vec{v}$

$\Rightarrow \left|\det\left[\vec{u} \mid \vec{v}\right]\right| =$ area of parallelog

Image parallelogram:

$L : \mathbb{R}^2 \to \mathbb{R}^2$

What is the area of the parallelogram induced by $L(\vec{x}), L(\vec{y})$

$$\text{area} = \left|\det\left[L(\vec{x}) \mid L(\vec{y})\right]\right|$$

$$= \left|\det [L]\left[\vec{x} \mid \vec{y}\right]\right|$$

$$= \left|\det([L])\right| \left|\det\left[\vec{x} \mid \vec{y}\right]\right|$$

$$= \left|\det([L])\right| \text{area}(\vec{x}, \vec{y}).$$

Example:

$$L: \mathbb{R}^2 \to \mathbb{R}^2$$

$$L(\vec{x}) = \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix} \quad [L] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\det [L] = -1$$

$$\vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{area}(\vec{x}, \vec{y}) = \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = 1$$

$$\Rightarrow \text{area}(L(\vec{x}), L(\vec{y}))$$

$$= \text{area}\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = |-1| \cdot 1 = 1$$

or

$$\text{area}\left(\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left|\det\begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\right|$$

$$= |-1| = 1$$

$\square$

Earlier:

Volume of parallelipiped $(\vec{u}, \vec{v}, \vec{w})$

$$= \left|\det(\vec{u} \,|\, \vec{v} \,|\, \vec{w})\right|.$$