Introduction of Bases

Review:

- Linear combination

$$\vec{a} = c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n$$

- Linear independence

$\vec{e}_1, \dots, \vec{e}_n$ are l.i. if

"$c_1\vec{e}_1 + c_2\vec{e}_2 + \dots + c_n\vec{e}_n = \vec{0}$

$\Rightarrow c_1 = c_2 = \dots c_n = 0$"

Ex.:

- $\vec{e}_1\ \vec{0}$
-  collinear
-  not collinear
- $\vec{w} = \vec{u} + 2\vec{v}$

  $\vec{u}, \vec{v}, \vec{w}$

- Span$(\vec{e})$ = line $(\vec{e} \neq \vec{0})$
- Span$(\vec{e}_1, \vec{e}_2)$ = plane $(\vec{e}_1, \vec{e}_2$ not collinear$)$
- Span$(\vec{e}_1, \vec{e}_2)$ = line $(\vec{e}_1, \vec{e}_2$ collinear$)$

Definition:

A basis of a line, plane, etc. is is a set of linearly independent vectors spanning the line, plane, etc.

Examples

- $\vec{e} \neq \vec{0}$ span$(\vec{e})$ = line = ℓ

  $\mathcal{B} = \{\vec{e}\}$ basis of ℓ

- $\vec{e}_1, \vec{e}_2$ not collinear $\rightarrow$ l.i.

  $\{\vec{e}_1, \vec{e}_2\}$ is a basis for span$(\vec{e}_1, \vec{e}_2)$ = plane

- $\vec{e}_1, \vec{e}_2$ collinear (not lin. ind.)

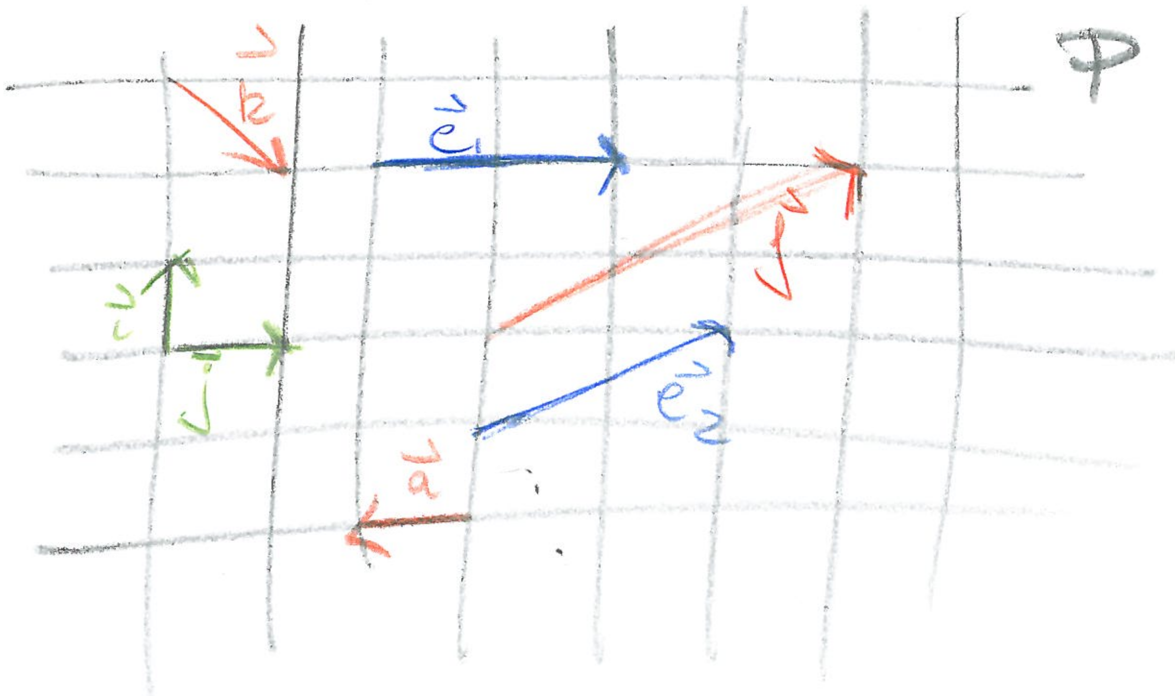
  $\Rightarrow$ $\mathcal{B} = \{\vec{e}_1, \vec{e}_2\}$ can not be a basis for a plane (span$(\mathcal{B})$ = line)

Remark:

$\mathcal{B} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis for a line, plane, etc.

then all vectors $\vec{a}$ in the line, plane, etc can be written as

$$\vec{a} = c_1\vec{e} + c_2\vec{e}_2 + \ldots + c_n\vec{e}_n$$



List bases for plane P

Write $\vec{b}$ as a lin. comb. of $\{\vec{i}, \vec{j}\}$ $\{\vec{e}_1, \vec{e}_2\}$

R.: Therefore we can identify, once a basis has been chosen

geometric vectors in the plane spanned by the basis with

all pairs of numbers $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ (call this $\mathbb{R}^2$)

- remember the example different bases give different coordinates to a given vector.

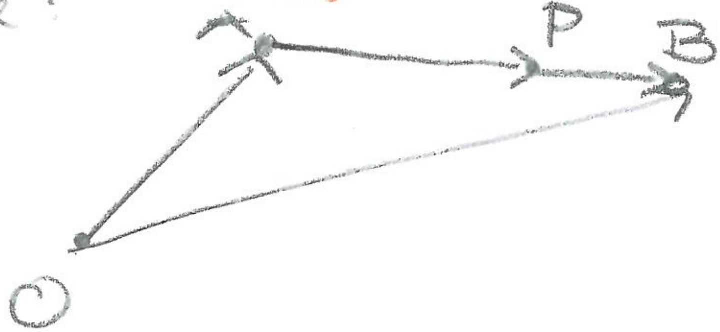
Introduction of a Coordinate System

Choose an origin O, move the initial points of the basis into O.

Now every point in the plane

can be "identified" by a location vector from O to the point which can be identified by its coordinates w.r.t. the basis - Cartesian coordinate system = Plane of Points

Example:



$A = (2,0)$

$B = (8,3)$

Find the coordinates of P, the point which divides the line segment between AB 2 : 1, i.e. AP : PB = 2 : 1

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$$

$$\overrightarrow{OA} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad \overrightarrow{OB} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

Find $\overrightarrow{AB}$

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= \begin{bmatrix} 8 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 6 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

What are the coordinates of P in a coordinate system with origin B?

$$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$$
$$= \begin{bmatrix} 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 8 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Bases of $\mathbb{R}^2$

Let $\{\vec{e}_1, \vec{e}_2\}$ be a basis of $\mathbb{R}^2$ (maybe $\vec{e}_1 = \vec{i}$, $\vec{e}_2 = \vec{j}$)

then $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2$ and

$\vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2$

is a basis if

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are linear independent. (basis does not matter)

Ex: TWO vectors are linear dependent if one is a multiple of the other. (collinear)

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \end{bmatrix}$
- $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix}$

Do the vectors span $\mathbb{R}^2$?

I.e. can we write every vector in $\mathbb{R}^2$ as a lin. comb. in those vectors? $\vec{a} \in \mathbb{R}^2$

Can we find $x_1, x_2 \in \mathbb{R}$ so that

1 $$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$1 \cdot x_1 + 0 \cdot x_2 = a_1$$
$$0 \cdot x_1 + 1 \cdot x_2 = a_2$$

$x_1 = a_1$
$x_2 = a_2$
is a solution

so $\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} = \mathbb{R}^2$

② $x_1\begin{bmatrix}1\\2\end{bmatrix} + x_2\begin{bmatrix}-3\\-6\end{bmatrix} = \begin{bmatrix}a_1\\a_2\end{bmatrix}$

$$\left[\begin{array}{l} x_1 - 3x_2 = a_1 \\ 2x_1 - 6x_2 = a_2 \end{array}\right. \qquad R_2 - 2R_1$$

$$\rightarrow \left[\begin{array}{l} x_1 - 3x_2 = a_1 \\ 2x_1 - 2x_1 - 6x_2 + 6x_2 = a_2 - 2a_1 \end{array}\right.$$

$$\rightarrow \left[\begin{array}{l} x_1 - 3x_2 = a_1 \\ \qquad 0 = a_2 - 2a_1 \end{array}\right.$$

This has only a solution if $a_2 - 2a_1 = 0$, not for all $\vec{a}$, $\operatorname{span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\-6\end{bmatrix}\right) \neq \mathbb{R}^2$

The space of geometric vectors on $\mathbb{R}^3$

Theorem:
Three vectors are linearly dependent if and only if they are coplanar.

$\rightarrow$ Three non coplanar vectors are linearly independent.

Theorem: Every vector in the space of geometric vectors can be written as a lin. comb. of three noncoplanar vectors.

$\rightarrow$ 3 noncoplanar vectors span the space of geometric vectors.

$\rightarrow$

Result: Three noncoplanar vectors form a basis of the space of geometric vectors.

Again we can for a given basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ identify a vector $\vec{a}$ with its components $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_B$ where $\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$

$\rightarrow \mathbb{R}^3$

Def.: $\left\{ \overset{\vec{i}}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{\vec{j}}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \overset{\vec{k}}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$ is the standard basis of $\mathbb{R}^3$.

Determine if three vectors are linearly independent (see $\mathbb{R}^2$)

Remember: Three vectors , $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are linearly independent if from

$$x_1 \vec{e}_1 + x_2 \vec{e}_2 + x_3 \vec{e}_3 = \vec{0}$$

follows that $x_1 = x_2 = x_3 = 0$

Example:

a) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ the vectors are not l. i.

b) $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

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$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 + 2x_2 + 2x_3 = 0 \\ 2x_1 \quad\quad + x_3 = 0 \end{cases}$$

Gaussian Algorithm
(eliminate x_1 from 2^{nd} and 3^{rd} row, then eliminate x_2 from 2^{nd} row)

$R_2 - R_1$, $R_3 - 2R_1$

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ \quad\; x_2 + x_3 = 0 \\ -2x_2 \quad\quad = 0 \end{cases} \qquad R_3 + 2R_2$$

$$\left.\begin{cases} x_1 + x_2 + x_3 = 0 \\ \quad\; x_2 + x_3 = 0 \\ \quad\quad\quad 2x_3 = 0 \end{cases}\right\} \begin{array}{l} R_3 \Rightarrow x_3 = 0 \\ \Rightarrow R_2 : x_2 = 0 \\ \Rightarrow x_1 = 0 \end{array}$$

$\Rightarrow$ vectors are l.i.

c) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-4\\-2\end{bmatrix}$

$R_3 - \frac{1}{2}R_2$

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_2 - 4x_3 = 0 \\ x_2 - 2x_3 = 0 \end{cases} \quad \begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_2 - 4x_3 = 0 \\ 0 = 0 \end{cases}$$

$\Rightarrow$ 2 equations 3 unknown
can choose one variable free

$2x_2 = 4x_3 \Rightarrow x_2 = 2x_3$,

$x_3 = t$, $t \in \mathbb{R}$

$\Rightarrow x_2 = 2t$

$x_1 + 2t - t = 0 \Rightarrow x_1 = -t$

Choose $t = 1$

then $\vec{x} = \begin{bmatrix}-1\\2\\1\end{bmatrix}$ is a solution

$\vec{x} \neq \vec{0} \Rightarrow$ vector are not linear indep.

d) find components of

$$\vec{a} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix} \text{ w.r.t. } B = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + a_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

$$\begin{cases} a_1 + a_2 + 2a_3 = 2 \\ a_1 + 2a_2 = -2 \\ a_1 + 2a_2 + a_3 = -1 \end{cases} \qquad \begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$$

$$\begin{cases} a_1 + a_2 + 2a_3 = 2 \\ a_2 - 2a_3 = -4 \\ a_2 - a_3 = -3 \end{cases} \qquad R_3 - R_2$$

$$\begin{cases} a_1 + a_2 + 2a_3 = 2 \\ a_2 - 2a_3 = -4 \\ a_3 = 1 \end{cases} \qquad \begin{matrix} a_3 = 1 \to R_2 \\ a_2 - 2 = -4 \\ a_2 = -2 \end{matrix}$$

$$a_2 = -2,\ a_3 = 1 \to R_1$$

$$a_1 + (-2) + 2 = 2 \to a_1 = 2$$

$$\to \vec{a} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}_B$$