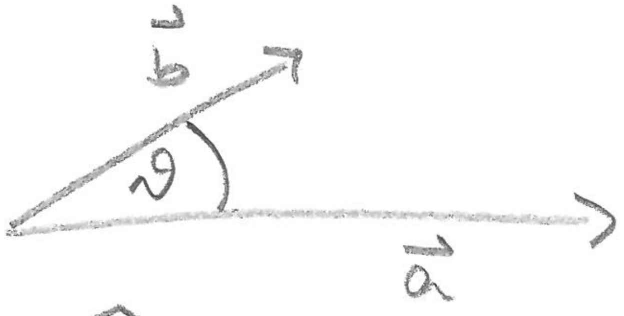
Dot Product

Multiply 2 vectors and get a scalar (number)!

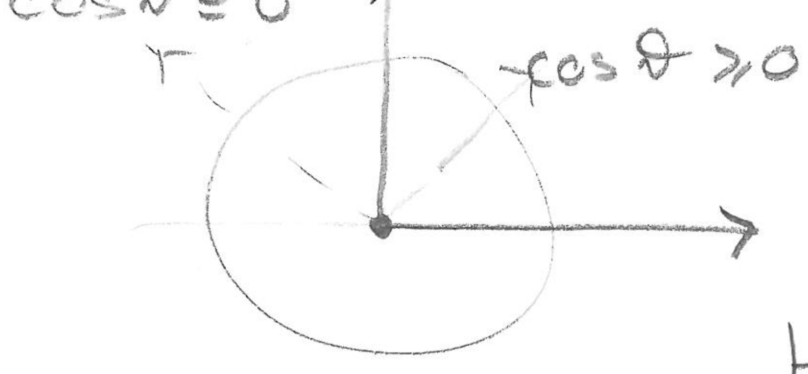


$\vartheta = \widehat{\vec{a}, \vec{b}}$ = smallest angle between $\vec{a}$ and $\vec{b}$ $\quad \vartheta \in [0, 180°]$ ($[0, \pi]$)

Definition:

The dot product of vectors $\vec{a}, \vec{b}$ is defined as

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos(\vartheta)$$



Perpendicular

$\vartheta = 90°$

$\cos(\vartheta) = 0$

2

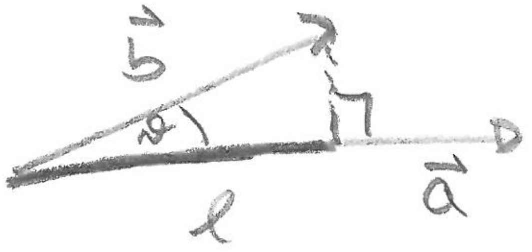
Theorem:

$$\vec{a} \cdot \vec{b} = 0 \iff \vartheta = 90^\circ$$
$$(\vec{a} \perp \vec{b})$$

- $\vec{a} = \vec{b}$

$$\vec{a} \cdot \vec{a} = \|\vec{a}\| \cdot \|\vec{a}\| \underbrace{\cos(0)}_{=1}$$
$$= \|\vec{a}\|^2$$
$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}}$$

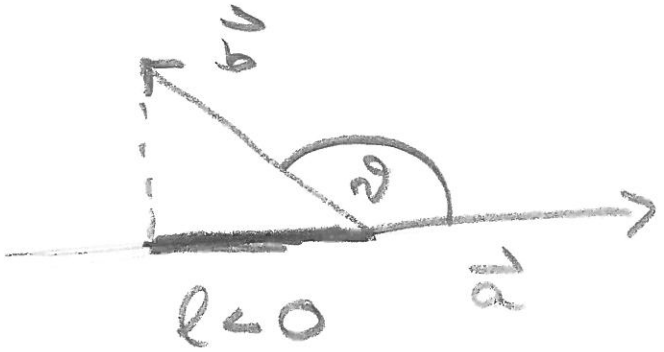
Projections:



$\ell = \text{proj}_{\vec{a}} \vec{b}$ = scalar projection

$$\cos\vartheta = \frac{\ell}{\|\vec{b}\|}$$

$$\Rightarrow \ell = \|\vec{b}\| \cdot \cos\vartheta \, \frac{\|\vec{a}\|}{\|\vec{a}\|} = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|}$$



Unit Vector

$\vec{u}$ is called a unit vector if it has length 1, $\|\vec{u}\| = 1$

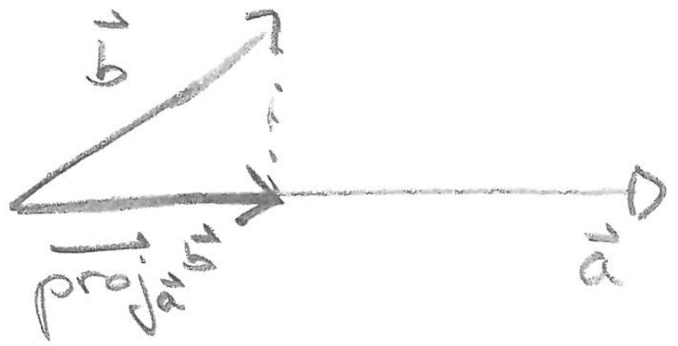
- $\vec{a}$ is a vector

$\to$ $\dfrac{\vec{a}}{\|\vec{a}\|}$ is a unit vector

$$\left(\left\| \frac{\vec{a}}{\|\vec{a}\|} \right\| = \frac{1}{\|\vec{a}\|} \|\vec{a}\| = 1 \right)$$

in the same direction as $\vec{a}$.

Vector projection



The vector projection of $\vec{b}$ on $\vec{a}$ is the vector with length $\mathrm{proj}_{\vec{a}}\vec{b}$ (scalar)

- codirected with $\vec{a}$ if $\theta < 90°$
- opposite directed with $\vec{a}$ if $\theta > 90°$

$$\overrightarrow{\mathrm{proj}}_{\vec{a}}\vec{b} = \mathrm{proj}_{\vec{a}}\vec{b}\ \frac{a}{\|\vec{a}\|}$$

$$= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|^2}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a}$$

Properties:

(1) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

(2) $(k \cdot \vec{a}) \cdot \vec{b} = k(\vec{a} \cdot \vec{b})$

(3) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

(4) $\vec{a} \cdot \vec{a} \geq 0$ with equality if $\vec{a} = \vec{0}$

$\vec{a}, \vec{b}$ orthogonal (perpendicular) $\Leftrightarrow \vec{a} \cdot \vec{b} = 0$

A basis is called orthonormal if

(1) all vectors are unit vectors

(2) vectors are orthogonal to each other

(basis is orthogonal, if only (2) holds)

Let $\{\vec{i}, \vec{j}, \vec{k}\}$ orthonormal basis

$$\vec{a} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$$

$$\vec{b} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$$

$$\Rightarrow \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$\mathbb{R}^3$ $\quad \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

orthonormal

Ex.: Find a unit vector in direction of $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3}$$

$$= \sqrt{14}$$

$$\vec{u} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Example: $B = \{\vec{i}, \vec{j}, \vec{k}\}$

$$\vec{a} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}_B \qquad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_B$$

- find the vector projection of $\vec{a}$ onto $\vec{a} - \vec{b}$

$$\overrightarrow{\text{proj}}_{\vec{a}-\vec{b}}\, \vec{a} = \frac{\vec{a} \cdot (\vec{a} - \vec{b})}{(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})} (\vec{a} - \vec{b})$$

$$\vec{a} - \vec{b} = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}_B$$

$$\vec{a} \cdot (\vec{a} - \vec{b}) = 1 \cdot 0 + (-2)(-2) + 2 \cdot 1$$
$$= 6$$

$$(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = 0 \cdot 0 + (-2)(-2) + 1 \cdot 1$$
$$= 5$$

$$\overrightarrow{\text{proj}}_{\vec{a}-\vec{b}}\, \vec{a} = \frac{6}{5} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

- find the angle between $\vec{a}$ and $\vec{b}$

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \cdot \|\vec{b}\| \cdot \cos\vartheta$$

$$\rightarrow \cos\vartheta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\| \, \|\vec{b}\|}$$

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot 1 + (-2) 0 + 2 \cdot 1$$

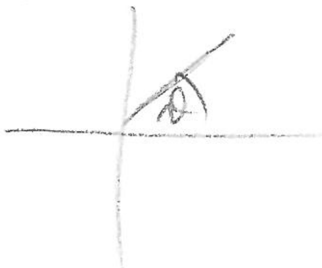
$$= 3$$

$$\|\vec{a}\| = \sqrt{1+4+4} = \sqrt{9} = 3$$

$$\|\vec{b}\| = \sqrt{1+0+1} = \sqrt{2}$$

$$\rightarrow \cos\vartheta = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$

$$\vartheta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$



$$= 45^\circ \ (\approx \frac{\pi}{2})$$