Cross Product

→ only defined for $\mathbb{R}^3$

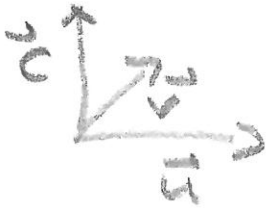
Definition.

$\vec{u}, \vec{v} \in \mathbb{R}^3$

The cross product, $\vec{u} \times \vec{w}$, is the vector $\vec{c}$ with

i) $\vec{c}$ is orthogonal to $\vec{u}$ <u>and</u> $\vec{v}$ (orthogonal span $(\vec{u}, \vec{v})$)

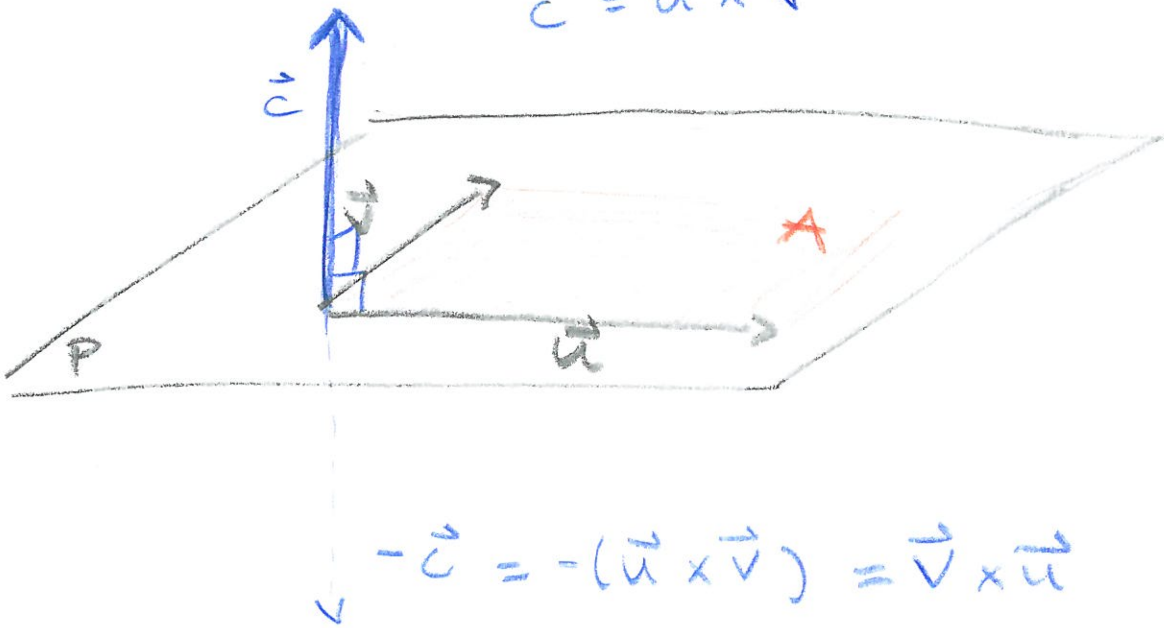
ii) $\vec{u}, \vec{v}, \vec{c}$ obey the right-hand rule.



iii) $\|\vec{c}\| = A$ the area of the parallelogram defined by $\vec{u}, \vec{v}$

$A = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$

$$\vec{c} = \vec{u} \times \vec{v}$$



$$-\vec{c} = -(\vec{u} \times \vec{v}) = \vec{v} \times \vec{u}$$

Properties:

(1) $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$

(2) $\alpha \vec{u} \times \vec{v} = \alpha(\vec{u} \times \vec{v})$

(3) $\vec{v} \times \vec{v} = \vec{0}$

(4) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$$\underline{\underline{\vec{u} \times \alpha \vec{v}}} \underset{(1)}{=} -(\alpha \vec{v} \times \vec{u}) \overset{\text{scalar prod.}}{\underset{(2)}{=}} (-1)(\alpha)(\vec{v} \times \vec{u})$$

$$\underset{\text{scalar prod.}}{=} \alpha\left(-(\vec{v} \times \vec{u})\right) \underset{(1)}{=} \underline{\underline{\alpha(\vec{u} \times \vec{v})}}$$

Calculation of the cross - product: 3

2x2 determinant

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

3x3 determinant

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Ex.: $\begin{vmatrix} 1 & 1 \\ -2 & 0 \end{vmatrix} = 1\cdot 0 - (-2)\cdot 1 = 2$

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & -2 & 3 \end{vmatrix} = 1\begin{vmatrix} 1 & 0 \\ -2 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & 0 \\ -1 & 3 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ -1 & -2 \end{vmatrix}$$

$$= 1(3-0) - 2(0-0) + 3(0-(-1))$$

$$= 3 + 3 = 6$$

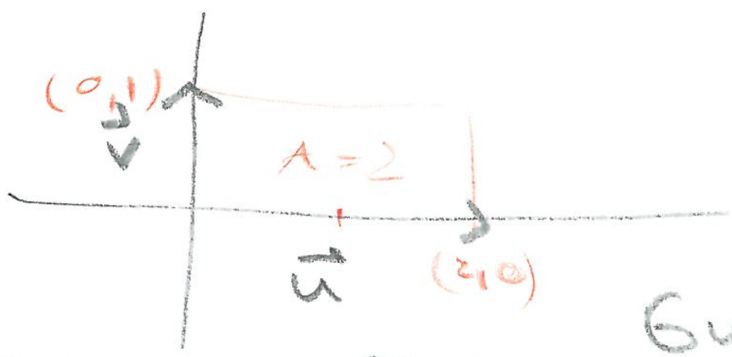
Let $B = \{\vec{i}, \vec{j}, \vec{k}\}$

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}_B \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_B$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

Ex.:



Guess:

$$\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{c} = 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \vec{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \vec{i} \cdot 0 + \vec{j} \cdot 0 + \vec{k} \cdot 2$$

Applications

1.) Finding the area of a parallelogram

2.) Finding the area of a triangle.

$$\triangle = \frac{1}{2}\ ▱$$

3.) $\vec{a} \cdot (\vec{b} \times \vec{c})$ = volume of the parallelepiped if $\vec{a}, \vec{b}, \vec{c}$ follow right hand rule

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \text{triple product of } \vec{a}, \vec{b}, \vec{c}$$

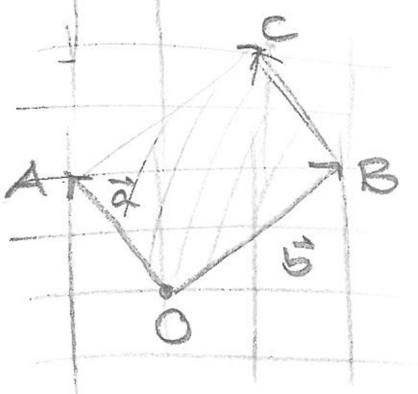
Theorem:

Three vectors are coplanar exactly when their triple product is 0.

Application examples:

1) Find the area of parallelogram defined by

$\vec{a} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$



$A = \|\vec{a} \times \vec{b}\| =$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 2 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 2 \\ 2 & 2 \end{vmatrix} \vec{k}$$

$$= 0 \cdot \vec{i} - 0\vec{j} + (-6)\vec{k} = \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}$$

$$\|\vec{a} \times \vec{b}\| = \sqrt{\begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix}} = \sqrt{0^2 + 0^2 + (-6)^2}$$

$$= 6$$

2) Find the area of the triangle OCB.

The area is half of the area of the parallelogram $\rightarrow 3$

3) Find the volume of the parallelepiped given by

$$\vec{a} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$|\vec{c} \cdot (\vec{a} \times \vec{b})| = \left| \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -6 \end{bmatrix} \right|$$

$$= |1 \cdot 0 + 1 \cdot 0 + 1 \cdot (-6)| = |-6| = 6$$

4) Are vectors

$$\vec{u} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

coplanar? Find the triple product and check, if it is 0!

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 2 & 1 \\ 2 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} \vec{k}$$

$$= (-1)\vec{i} - (-2)\vec{j} + 4\vec{k} = \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = 1 - 2 + 12$$

$$= \underline{\underline{11}}$$

Volume = 11