Some Applications of Solving Lin. Equations

- Spanning

$\mathrm{Span}(\vec{v}_1, \ldots, \vec{v}_n)$

= set of all linear combinations in $\vec{v}_1, \ldots, \vec{v}_n$

Ex.: Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the span

of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$?

Do $t_1, t_2 \in \mathbb{R}$ exist such that

$$t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} ?$$

$$\Leftrightarrow \quad \begin{aligned} t_1 + 2 \cdot t_2 &= 2 \\ t_2 &= 2 \\ 2t_2 &= 3 \end{aligned} \quad \rightarrow \text{solve the system}$$

Ex.:

Do $\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$ span $\mathbb{R}^3$?

then for all $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

$$t_1 \begin{bmatrix}1\\2\\3\end{bmatrix} + t_2 \begin{bmatrix}1\\1\\1\end{bmatrix} + t_3 \begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

has a solution!

This is exactly true

when $\text{rank}\begin{pmatrix}1 & 1 & 1\\2 & 1 & 0\\3 & 1 & 0\end{pmatrix} = 3$

number of variables

(see theorem before)

i.e. the lin. systems has a solution for all $\vec{x}$.

Conclusion:

A set of vectors $\{\vec{v}_1, \dots, \vec{v}_n\}$ spans $\mathbb{R}^n$ if and only if with

$A = (\vec{v}_1 | \vec{v}_2 | \dots | \vec{v}_n)$: $\text{rank}(A) = n$

Linear Independence

Vectors $\vec{v}_1, \dots \vec{v}_n$ are lin. indep

if and only if

from

$$(*)\ t_1\vec{v}_1 + \dots + t_n\vec{v}_n = \vec{0}$$

it follows that

$t_1 = \dots = t_n = 0$ (trivial sol.)

if and only if

$(*)$ has exactly one solution

if and only if

$$\operatorname{rank}\left[\vec{v}_1 \,\middle|\, \vec{v}_2 \cdots \middle|\, \vec{v}_n\right] = n$$

Ex. 1

Are $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\7\end{bmatrix}$ lin. indep.?

$$\begin{bmatrix}1 & 1 & 3\\1 & 0 & -1\\1 & 2 & 7\end{bmatrix} \xrightarrow[R_3-R_1]{R_2-R_1} \begin{bmatrix}1 & 1 & 3\\0 & -1 & -4\\0 & 1 & 4\end{bmatrix} \xrightarrow{R_3+R_2}$$

$$\begin{bmatrix}1 & 1 & 3\\0 & -1 & -4\\0 & 0 & 0\end{bmatrix}$$

pivots

rank $= 2 \neq 3$

$\longrightarrow$ not lin. indep.

Theorem:

Let $\{\vec{v}_1, \ldots \vec{v}_k\} \in \mathbb{R}^n$.

If $\operatorname{span}(\{\vec{v}_1, \ldots, \vec{v}_k\}) = \mathbb{R}^n$ then $k = n$.