0.1 Linear Transformations

A *function* is a rule that assigns a value from a set B for each element in a set A.
Notation: $f : A \mapsto B$
If the value $b \in B$ is assigned to value $a \in A$, then write $f(a) = b$, b is called the *image* of a under f.
A is called the *domain* of f and B is called the *codomain.*
The subset of B consisting of all possible values of f as a varies in the domain is called the *range* of f.

Definition 1
Two functions f_1 and f_2 are called equal, if their domains are equal and

$$f_1(a) = f_2(a) \text{ for all } a \text{ in the domain}$$

Example 1

function	example	description
$f(x)$	$f(x) = x - 2$	Function from $\mathbb{R}$ to $\mathbb{R}$
$f(x, y)$	$f(x, y) = x + y$	Function from $\mathbb{R}^2$ to $\mathbb{R}$
$f(x, y, z)$	$f(x, y, z) = x + y + z$	Function from $\mathbb{R}^3$ to $\mathbb{R}$
$f(x, y, z)$	$f(x, y, z) = (x + y, z)$	Function from $\mathbb{R}^3$ to $\mathbb{R}^2$

Functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ If $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, then f is called a *map* or a *transformation.*
If $m = n$, then f is called an operator on $\mathbb{R}^n$.
Let $f_1, f_2, \dots f_m$ functions from $\mathbb{R}^n$ to $\mathbb{R}$, assume

$$\begin{gathered} f_1(x_1, x_2, \dots, x_n) = w_1 \\ f_2(x_1, x_2, \dots, x_n) = w_2 \\ \vdots \\ f_m(x_1, x_2, \dots, x_n) = w_m \end{gathered}$$

then the point $(w_1, w_2, \dots, w_m) \in \mathbb{R}^m$ is assigned to $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and thus those functions define a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
Denote the transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ and

$$T(x_1, x_2, \dots, x_n) = (w_1, w_2, \dots, w_m)$$

Example 2
$f_1(x_1, x_2) = x_1 + x_2, f_2(x_1, x_2) = x_1 x_2$ define an operator $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$.

$$T(x_1, x_2) = (x_1 + x_2, x_1 x_2)$$

Linear Transformations In the special case where the functions $f_1, f_2, \dots, f_m$ are linear , the transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is called a linear transformation.

A linear transformation is defined by equations

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & w_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & w_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & w_m \end{array}$$

or in matrix notation

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix}$$

or

$$A\mathbf{x} = \mathbf{w}$$

The matrix A is called the *standard matrix for the linear transformation* T, and T is called *multiplication by* A.

Remark:
Through this discussion we showed that a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ correspond to matrices of size $m \times n$.
One can say that to each matrix A there corresponds a linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$, and to each linear $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ transformation there corresponds an $m \times n$ matrix A.

Example 3
Let $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ defined by

$$\begin{array}{ccccccc} 2x_1 & + & 3x_2 & + & (-1)x_3 & = & w_1 \\ x_1 & + & x_2 & + & (-1)x_3 & = & w_2 \end{array}$$

can be expressed in matrix form as

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

The standard matrix for T is

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

The image of a point (x_1, x_2, x_3) can be found by using the defining equations or by matrix multiplication.

$$T(1, 2, 0) = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$

Notation:
If $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a multiplication by A, and if it important to emphasize the standard matrix then we shall denote the transformation by $T_A : \mathbb{R}^n \mapsto \mathbb{R}^m$. Thus

$$T_A(\mathbf{x}) = A\mathbf{x}$$

Since linear transformations can be identified with their standard matrices we will use $[T]$ as symbol for the standard matrix for $T : \mathbb{R}^n \mapsto \mathbb{R}^m$.

$$T(\mathbf{x}) = [T]\mathbf{x} \text{ or } [T_A] = A$$

Geometry of linear Transformations
A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ transforms points in $\mathbb{R}^n$ into new points in $\mathbb{R}^m$

Example 4
Zero Transformation The zero transformation from $T_0 : \mathbb{R}^n \mapsto \mathbb{R}^m$ has standard matrix 0, so that

$$T_0(\mathbf{x}) = \mathbf{0}$$

for all $\mathbf{x} \in \mathbb{R}^n$

Example 5
Identity Transformation The identity transformation $T_I : \mathbb{R}^n \mapsto \mathbb{R}^m$ has standard matrix I_n ($n \times n$ identity matrix), so that

$$T_I(\mathbf{x}) = I_n\mathbf{x} = \mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Among the more important transformations are those that cause reflections, projections, and rotations

Example 6
Reflections
Consider $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with standard matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$T(\mathbf{x}) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix}$$

T reflects points (x_1, x_2) about the y-axis.
What might be the standard matrix of the linear transformation reflecting point about the x-axis?

$\mathbb{R}^2 \mapsto \mathbb{R}^2$

Operator	Equation	Standard matrix
Reflection about the y-axis	$T(x,y)=(-x,y)$	$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
Reflection about the x-axis	$T(x,y)=(x,-y)$	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Reflection about the line $y=x$	$T(x,y)=(y,x)$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\mathbb{R}^3 \mapsto \mathbb{R}^3$

Operator	Equation	Standard matrix
Reflection about the xy-plane	$T(x,y,z)=(x,y,-z)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$
Reflection about the xz-plane	$T(x,y,z)=(x,-y,z)$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
Reflection about the yz-planes	$T(x,y,z)=(-x,y,z)$	$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 7

Projections Consider $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with standard matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

then

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

It gives the orthogonal projection of point (x,y) onto the x-axis.

Consider $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ with standard matrix

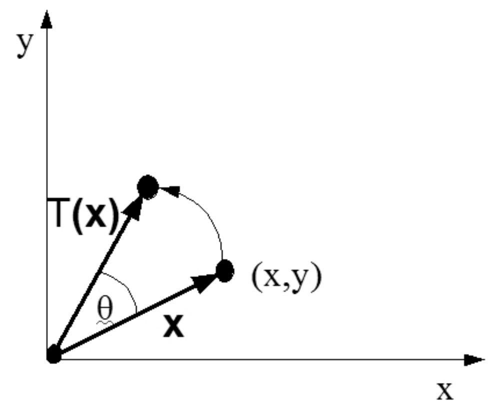
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$$

It gives the orthogonal projection of a point (x,y,z) onto the xy-plane.

Example 8
Rotation:
An operator that rotates a vector in $\mathbb{R}^2$ through a given angle θ is called a rotation operator in $\mathbb{R}^2$.

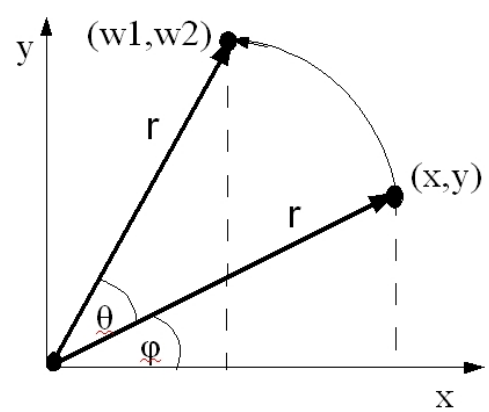


$$T_R(\mathbf{x}) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

The standard matrix of a rotation operator in $\mathbb{R}^2$ for angle θ is therefore

$$[T_R] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Proof:



Let $(w_1, w_2) = T_R(\mathbf{x})$, then (check the diagram)

$$w_1 = r\cos(\theta + \varphi), \quad w_2 = r\sin(\theta + \varphi)$$

Using trigonometric identities

$$w_1 = r\cos(\theta)\cos(\varphi) - r\sin(\theta)\sin(\varphi)$$
$$w_2 = r\sin(\theta)\cos(\varphi) + r\cos(\theta)\sin(\varphi)$$

Also (check diagram)

$$x = r\cos\varphi, \quad y = r\sin\varphi$$

substituting the later into the equations above gives

$$w_1 = x\cos(\theta) - y\sin(\theta), \quad w_2 = x\sin(\theta) + y\cos(\theta)$$

therefore

$$T_R(x,y) = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 9

The standard matrix of the rotation by $\pi/2$ is

$$[T_R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

therefore

$$T_R(1,2) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

The standard matrix of the rotation by $\pi/4$ is

$$[T_R] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

therefore

$$T_R(1,2) = \begin{bmatrix} -1/\sqrt{2} \\ 3/\sqrt{2} \end{bmatrix}$$

Rotation in $\mathbb{R}^3$

Operator	Equation	Standard matrix
Counterclockwise rotation about the positive x-axis through an angle θ	$T(x,y,z) = \begin{bmatrix} x \\ y\cos\theta - z\sin\theta \\ y\sin\theta + z\cos\theta \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive y-axis through an angle θ	$T(x,y,z) = \begin{bmatrix} x\cos\theta + z\sin\theta \\ y \\ -x\sin\theta + z\cos\theta \end{bmatrix}$	$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$
Counterclockwise rotation about the positive z-axis through an angle θ	$T(x,y,z) = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \end{bmatrix}$	$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Dilation and Contraction

This is the operator stretching or shrinking a vector by a factor k, but keeping the direction unchanged. We call the operator a dilation if the transformed vector is at least as long as the original vector, and a contraction if the transformed vector is at most as long as the original vector.

Operator	Equation	Standard matrix
Contraction with factor k on $\mathbb{R}^2$, $0 \leqslant k \leqslant 1$)	$T(x, y) = \begin{bmatrix} kx \\ ky \end{bmatrix}$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$
Dilation with factor k on $\mathbb{R}^2$, $k \geqslant 1$)	$T(x, y) = \begin{bmatrix} kx \\ ky \end{bmatrix}$	$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Operator	Equation	Standard matrix
Contraction with factor k on $\mathbb{R}^3$, $0 \leqslant k \leqslant 1$)	$T(x, y, z) = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$	$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$
Dilation with factor k on $\mathbb{R}^3$, $k \geqslant 1$)	$T(x, y, z) = \begin{bmatrix} kx \\ ky \\ kz \end{bmatrix}$	$\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$

Composition of Linear Transformations

Let $T_A : \mathbb{R}^n \mapsto \mathbb{R}^k$ and $T_B : \mathbb{R}^k \mapsto \mathbb{R}^m$ be linear transformations, then for each $\mathbf{x} \in \mathbb{R}^n$ one can first compute $T_A(\mathbf{x})$, which is a vector in $\mathbb{R}^k$ and then one can compute $T_B(T_A(\mathbf{x}))$, which is a vector in $\mathbb{R}^m$.

Thus the application of first T_A and then of T_B is a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. It is called the composition of T_B with T_A. and is denoted as $T_B \circ T_A$ (read T_B circle T_A).

$$(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x})) = T_B(A\mathbf{x}) = B(A\mathbf{x}) = BA\mathbf{x}$$

Therefore the standard matrix of the composition of T_B with T_A is BA.

$$T_B \circ T_A = T_{BA}$$

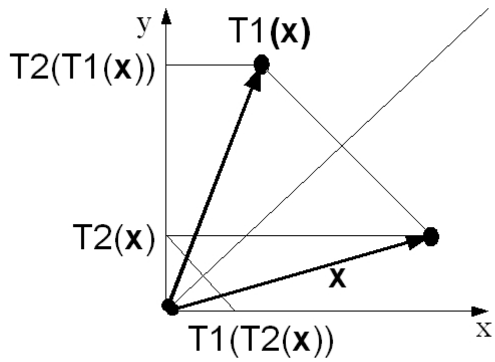
Remark:

This equation points out an important interpretation of the matrix product. Composition of two linear transformations is equivalent to the multiplication of two matrices.

Example 10

In general: Composition is not commutative.

T_1:reflection about $y = x$, and T_2 orthogonal projection onto y



One can easily generalize the concept to the composition of more than two transformations.

0.2 Properties of linear Transformations

One-to-One Linear Transformations
Transformations that transform different vectors into different images, that is
If $\mathbf{x} \neq \mathbf{y}$ therefore $T(\mathbf{x}) \neq T(\mathbf{y})$,
are of special interest.
One such example is the rotation by an angle θ in $\mathbb{R}^2$.
But the orthogonal projection onto the $xy - plane$ in $\mathbb{R}^3$ does not have this property.
$T(x_1, x_2, x_3) = (x_1, x_2, 0)$, so $T(2, 1, 1) = T(2, 1, 45)$.

Definition 2
A linear transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is said to be one-to-one, if it is true that

$$\mathbf{x} \neq \mathbf{y} \Rightarrow T(\mathbf{x}) \neq T(\mathbf{y})$$

distinct vectors in $\mathbb{R}^n$ are mapped into distinct vectors in $\mathbb{R}^m$.

Conclusion:
If T is one-to-one and $\mathbf{w}$ is a vector in the range of T, then there is exactly one vector in $\mathbb{R}^n$ with $T(\mathbf{x}) = \mathbf{w}$.

Consider transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the standard matrices are square matrices of size $n \times n$.

Theorem 1
If A is a $n \times n$ matrix and $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is the multiplication by A, then the following statements are equivalent

(a) A is invertible

(b) The range of T_A is $\mathbb{R}^n$

(c) T_A is one-to-one.

Proof:

(a)$\Rightarrow$(b)

(b)$\Rightarrow$(c)

(c)$\Rightarrow$(a)

Application:
The rotation by θ in $\mathbb{R}^2$ is one-to-one.
The standard matrix of this operator is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

then $det(A) = \cos^2\theta + \sin^2\theta = 1 \neq 0$, therefore A is invertible and therefore the rotation operator is one-to-one.

Show yourself using this criteria that the orthogonal projection in $\mathbb{R}^3$ is NOT one-to-one.

Inverse of a one-to-one Operator

Definition 3
If $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a one-to-one operator, then $T^{-1} = T_{A^{-1}}$ is called the inverse operator of T_A.

Example 11
Let $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ with

$$[T] = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$$

Then

$$T(x_1, x_2) = (x_1 - x_2, -2x_1) \text{ for } (x_1, x_2) \in \mathbb{R}^2$$

and since

$$[T]^{-1} = \frac{-1}{2}\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

and

$$T^{-1}(x_1, x_2) = (-x_2/2, -(2x_1 + x_2)/2).$$

Theorem 2
Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a one-to-one operator, then

(a)

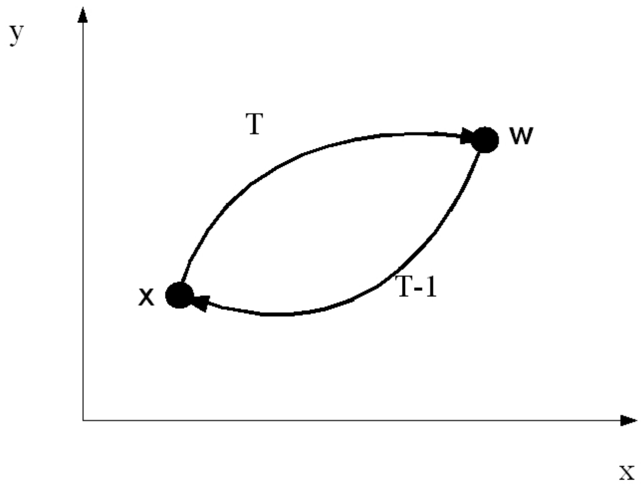
$$T(\mathbf{x}) = \mathbf{w} \Leftrightarrow T^{-1}(\mathbf{w}) = \mathbf{x}$$

(b) For $\mathbf{x} \in \mathbb{R}^n$ it is $(T \circ T^{-1})(\mathbf{x}) = \mathbf{x}$, and $(T^{-1} \circ T)(\mathbf{x}) = \mathbf{x}$

Proof:

(a) If $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a one-to-one operator and $T(\mathbf{x}) = \mathbf{w}$, then the standard matrix $[T]$ is invertible and

$$\begin{aligned} & T(\mathbf{x}) = \mathbf{w} \\ \Leftrightarrow\ & [T]\mathbf{x} = \mathbf{w} \\ \Leftrightarrow\ & [T]^{-1}[T]\mathbf{x} = [T]^{-1}\mathbf{w} \\ \Leftrightarrow\ & I_n\mathbf{x} = [T]^{-1}\mathbf{w} \\ \Leftrightarrow\ & \mathbf{x} = [T^{-1}]\mathbf{w} \\ \Leftrightarrow\ & \mathbf{x} = T^{-1}(\mathbf{w}) \end{aligned}$$



(b) If $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a one-to-one operator and $T(\mathbf{x}) = \mathbf{w}$, then the standard matrix A is invertible and

$$T \circ T^{-1} = T_A \circ T_{A^{-1}} = T_{AA^{-1}} = T_I$$

therefore the claim holds.

Theorem 3
A transformation $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear if and only if the following relationship holds for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. $T(c\mathbf{u}) = cT(\mathbf{u})$

Example 12
Show that

$$T(x_1, x_2) = (x_1 - x_2, -2x_1) \text{ for } (x_1, x_2) \in \mathbb{R}^2$$

is a linear transformation.

(a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$, then

$$T(\mathbf{u} + \mathbf{v}) = (u_1 + v_1 - (u_2 + v_2), -2(u_1 + v_1)) = (u_1 - u_2, -2u_1) + (v_1 - v_2, -2v_1) = T(\mathbf{u}) + T(\mathbf{v})$$

(b)

$$T(c\mathbf{u}) = (cu_1 - (cu_2)), -2(cu_1)) = c(u_1 - u_2, -2u_1) = cT(\mathbf{u})$$

Both properties hold, therefore T is a linear transformation.

Theorem 4
If $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation and $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ are the the standard basis vectors for $\mathbb{R}^n$, then the standard matrix for T is

$$[T] = \left[\ T(\mathbf{e_1})\ \middle|\ T(\mathbf{e_2})\ \middle|\ \ldots\ \middle|\ T(\mathbf{e_n})\ \right]$$

Example 13

Let T be the orthogonal projection onto the yz-plane in $\mathbb{R}^3$. Then

$$T(\mathbf{e_1}) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, T(\mathbf{e_2}) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, T(\mathbf{e_3}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore

$$[T] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$