1 General Vector Spaces

In the last chapter 2- and 3-space were generalized, and we saw that no new concepts arose by dealing with $\mathbb{R}^n$.
In a next step we want to generalize $\mathbb{R}^n$ to a general n-dimensional space, a vector space.

Definition 1
Let V be a non empty set on which two operations, addition($\oplus$) and scalar multiplication($\odot$), are defined.
If the following axioms are satisfied for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and all scalars $k, l \in \mathbb{R}$, then V is called a vector space, and the elements of V are called vectors.

1. $\mathbf{u} \oplus \mathbf{v} \in V$
2. $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$
3. $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$
4. There is an element $\mathbf{0} \in V$, called the zero vector for V, such that $\mathbf{0} \oplus \mathbf{u} = \mathbf{u} \oplus \mathbf{0} = \mathbf{u}$.
5. For each $\mathbf{u} \in V$ there is an element $-\mathbf{u} \in V$, called the negative of $\mathbf{u}$, such that $\mathbf{u} \oplus -\mathbf{u} = \mathbf{0}$
6. $k \odot \mathbf{u} \in V$
7. $k \odot (\mathbf{u} \oplus \mathbf{v}) = k \odot \mathbf{u} \oplus k \odot \mathbf{v}$
8. $(k + l) \odot \mathbf{u} = k \odot \mathbf{u} \oplus l \odot \mathbf{u}$
9. $k \odot (m \odot \mathbf{u}) = (km) \odot \mathbf{u}$
10. $1 \odot \mathbf{u} = \mathbf{u}$

Example 1
$\mathbb{R}^n$ is a vector space, with the addition and multiplication introduced in the last chapter.

Example 2
The set V of 2×2 matrices is a vector space using the matrix addition and matrix scalar multiplication.
To prove this statement all axioms have to be checked.
For the first axiom, we need to see if the sum of two 2×2 matrices is still a 2×2 matrix, and it is.
Matrix addition is commutative, so axiom 2 holds.
Similar axiom 3 holds.
By choosing the zero vector $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then axioms 4 hold.
The negative of a matrix $\mathbf{u} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, can be chosen to be $-\mathbf{u} = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$
Since the multiplication of a scalar and a 2×2 matrix is still a 2×2 matrix axiom 6 holds.
Prove in a similar way that all the other axioms hold, therefore the set of 2×2 matrices is a vector space.

Example 3
The set V of all $m \times n$ matrices is a vector space.

Example 4
Every plane through the origin is a vector space, with the standard vector addition and scalar multiplication.
(Every plane not including the origin is not a vector space.)
Again, we need to prove that all 10 axioms hold, to prove that this is true.
Why does the plane have to include the origin?

Example 5
Let V be a set of exactly one object, call this object $\mathbf{0}$, and define $\mathbf{0} + \mathbf{0} = \mathbf{0}$, and $k\mathbf{0} = \mathbf{0}$ for all $k \in \mathbb{R}$, then V is a vector space.

Example 6
The set V of all functions $f : \mathbb{R} \mapsto \mathbb{R}$ is a vector space.
If f and g are two such functions then $f + g$ is defined to be the function with

$$(f+g)(x) = f(x) + g(x) \quad for \quad x \in \mathbb{R}$$

and for $k \in \mathbb{R}$ the function kf is defined by

$$(kf)(x) = k\ f(x) \quad for \quad x \in \mathbb{R}$$

All 10 axioms have to be shown to hold true in order to establish that V is a vector space. This vector space is denoted by $F(-\infty, \infty)$.

Theorem 1
Let V be a vector space, $\mathbf{u} \in V$, and $k \in \mathbb{R}$, then

(a) $0\mathbf{u} = \mathbf{0}$

(b) $k\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{u} = -\mathbf{u}$

(d) If $k\mathbf{u} = \mathbf{0}$, then $k = 0$ or $\mathbf{u} = \mathbf{0}$

Proof:

(a) Assume $\mathbf{u} \in V$, then

$$0\mathbf{u} = (0+0)\mathbf{u} =_{axiom8} 0\mathbf{u} + 0\mathbf{u}$$

therefore

$$\begin{array}{lrcl} & 0\mathbf{u} + (-0\mathbf{u}) & = & 0\mathbf{u} + 0\mathbf{u} + (-0\mathbf{u}) \\ \Leftrightarrow \text{(axiom5)} & \mathbf{0} & = & 0\mathbf{u} + \mathbf{0} \\ \Leftrightarrow \text{(axiom5)} & \mathbf{0} & = & 0\mathbf{u} \end{array}$$

(d) Assume $\mathbf{u} \in V$ and $k \in \mathbb{R}, k \neq 0$, to prove (d), we have to show that from $k\mathbf{u} = \mathbf{0}$ follows $\mathbf{u} = \mathbf{0}$.

$$\begin{array}{rrcl} & \mathbf{0} & = & k\mathbf{u} \\ \Leftrightarrow & \frac{1}{k}\mathbf{0} & = & \frac{1}{k}(k\mathbf{u}) \\ \Leftrightarrow \text{((b), axiom9)} & \mathbf{0} & = & (\frac{1}{k}k)\mathbf{u} \\ & \mathbf{0} & = & 1\mathbf{u} \\ \Leftrightarrow \text{(axiom10)} & \mathbf{0} & = & \mathbf{u} \end{array}$$

2 Subspaces

Definition 2
A subset W of a vector space V is called a subspace of V, if W is a vector space under the addition and multiplication as defined on V.

Theorem 2
If W is a non empty subset of a vector space V, then W is a subspace of V if and only if the following conditions hold

1. If $\mathbf{u}, \mathbf{v} \in W$ then $\mathbf{u} + \mathbf{v} \in W$.

2. If $k \in \mathbb{R}$, and $\mathbf{u} \in W$, then $k\mathbf{u} \in W$.

Proof: text book

Example 7
In $\mathbb{R}^2$ lines through the origin are subspaces.
Geometrically we can see that the sum of two vectors on the line is still on the line, and a scalar multiple of a vector on the line is also on the line.

The line has to include the origin because otherwise $0\mathbf{u} = \mathbf{0}$ would not be on the line, and the line would not be a subspace.

Example 8
Any plane in $\mathbb{R}^3$ including the origin is a subspace of the vector space of $\mathbb{R}^3$.

Example 9
W the set of vectors in $\mathbb{R}^2$ with $x \geqslant 0$, and $y \geqslant 0$ is not a subspace, because $(1, 2) \in W$, but $(-2)(1, 2) = (-2, -4)$ is not in W.

Example 10
Polynomials of degree $\leqslant n$ is a subspace of $F(-\infty, \infty)$ the set of all real-valued functions. Let P_n denote the set of all such polynomials. Then $p \in P_n$ if

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

Then, if $p, q \in P_n$ with $q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$ then

$$\begin{aligned} p(x) + q(x) &= a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + b_0 + b_1x + b_2x^2 + \ldots + b_nx^n \\ &= (a_0 + b_0) + (a_1 + b_1)x + (a_+b_2)x^2 + \ldots + (a_n + b_n)x^n \end{aligned}$$

therefore $p + q \in P_n$. For $k \in \mathbb{R}$

$$kp(x) = k(a_0 + a_1x + a_2x^2 + \ldots + a_nx^n) = (ka_0) + (ka_1)x + (ka_2)x^2 + \ldots + (ka_n)x^n$$

therefore $kp \in P_n$. Since both properties from the theorem hold, we conclude that P_n is a subspace of $F(-\infty, \infty)$.

Example 11
The set of continuous function on $\mathbb{R}$ is a subspace of $F(-\infty, \infty)$ and denoted as $C(-\infty, \infty)$
The set of functions on $\mathbb{R}$ with continuous first derivatives is closed under addition and scalar multiplication and therefore a subspace of $F(-\infty, \infty)$ and denoted by $C^1(-\infty, \infty)$, and also a subspace of $C(-\infty, \infty)$.

Theorem 3
If $A\mathbf{x} = \mathbf{0}$ is a homogeneous linear system of m equations in n unknowns, then the set of solution vectors is a subspace of $\mathbb{R}^n$.

Proof:
Then W is not empty, because $\mathbf{0}$ is a solution of a homogeneous linear system and therefore in W.
Show that W is closed under addition and scalar multiplication.
Assume $\mathbf{x_1}$ and $\mathbf{x_2}$ are solutions of the linear system, then

$$\begin{aligned} A(\mathbf{x_1} + \mathbf{x_2}) &= A\mathbf{x_1} + A\mathbf{x_2} \\ &= \mathbf{0} + \mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

and

$$\begin{aligned} A(k\mathbf{x_1}) &= kA\mathbf{x_1} \\ &= k\mathbf{0} \\ &= \mathbf{0} \end{aligned}$$

which proves that $\mathbf{x_1} + \mathbf{x_2}$ and $k\mathbf{x_1}$ are in W.

Definition 3
A vector $\mathbf{w}$ is called a linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, if it can be expressed in the form

$$\mathbf{w} = k_1\mathbf{v_1} + k_2\mathbf{v_2} + \ldots + k_r\mathbf{v_r}$$

where $k_1, k_2, \ldots, k_r \in \mathbb{R}$.

Example 12
All vectors in $\mathbb{R}^3$ are linear combinations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

Example 13
Let $\mathbf{u} = (1, 1, 0)$, and $\mathbf{v} = (-1, 1, 0)$, then $\mathbf{w} = (5, 1, 0) = 3\mathbf{u} - 2\mathbf{v}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, but $\mathbf{x} = (1, 1, 1)$ is not a linear combination.
To show that $\mathbf{x}$ is not a linear combination, set up the following linear system

$$k_1\mathbf{u} + k_2\mathbf{v} = \mathbf{x}$$

If such numbers $k_1, k_2 \in \mathbb{R}$ can be found then $\mathbf{x}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, if no solution exists then $\mathbf{x}$ is NOT a linear combination of $\mathbf{u}$ and $\mathbf{v}$. The linear system is

$$\begin{aligned} k_1 - k_2 &= 1 \\ k_1 + k_2 &= 1 \\ 0 + 0 &= 1 \end{aligned}$$

Since the last equation is never true, this linear system has no solution, and $\mathbf{x}$ is not a linear combination of the $\mathbf{u}$ and $\mathbf{v}$.

Theorem 4
If V is a vector space, and $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r} \in V$, then

(a) The set W of all linear combinations of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ is a subspace in V.

(b) W is the smallest subspace of V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$, i.e. every subspace in V that contains $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ must contain W.

Proof: Show that the (a) and (b) from theorem 2 hold.

Definition 4
If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a set of vectors in the vector space V, then the subspace W of V consisting of all linear combinations of the vectors in S is called

the space spanned by $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$,

and we say that $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ span W.
We write $W = span(S)$ or $W = span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$.

Example 14
The polynomials of degree $\leqslant$ are spanned by $1, x, x^2, \ldots, x^n$ since each polynomial of degree $\leqslant n$ can be written as

$$p = a_0 + a_1 x + a_2 x2 + \ldots + a_n x^n$$

Example 15
One vector spans a line, two vectors where one is not a multiple of the other span a plane.

Theorem 5
If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ and $S' = \{\mathbf{w_1}, \mathbf{w_2}, \ldots, \mathbf{w_k}\}$ are two sets of vectors in a vector space V, then

$span(S) = span(S')$ if and only if each vector in S is a linear combination of a vector in $S'$and each vector in S' is a linear combination of vectors in S.

3 Linear Independence

Definition 5
If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a non empty set of vectors, then the vector equation

$$k_1\mathbf{v_1} + k_2\mathbf{v_2} + \ldots k_r\mathbf{v_r} = \mathbf{0}$$

has at least one solution $k_1 = k_2 = \ldots = k_r = 0$.
If this is the only solution then S is called linearly independent, otherwise S is called linearly dependent.

Example 16
Let $p, q \in P_3$, with $p(x) = 1 + x^2$ and $q(x) = x$, then p and q are linearly independent, because

$$k_1 p(x) + k_2 q(x) = k_1 + k_2 x + k_1 x^2 = 0 \quad \text{for all } x \in \mathbb{R}$$

only if $k_1 = k_2 = 0$.
But let $r \in P_n$ with $r(x) = 1 - x + x^2$, then the set $\{p, q, r\}$ is linearly dependent, because

$$\begin{aligned}
&k_1 p(x) + k_2 q(x) + k_3 r(x) = (k_1 + k_3) + (k_2 - k_3)x + (k_1 + k_3)x^2 = 0 \;\text{ for all } x \in \mathbb{R} \\
&\Leftrightarrow k_1 + k_3 = 0 \;\text{ and }\; k_2 - k_3 = 0 \;\text{ and }\; k_1 + k_3 = 0 \\
&\Leftrightarrow k_3 = t, k_2 = t, k_1 = -t \;\text{ for }\; t \in \mathbb{R}
\end{aligned}$$

Theorem 6
A set S with 2 or more vectors is

(a) linearly dependent if and only if at least one of the vectors in S is expressible as a linear combination of the other vectors in S.

(b) linearly independent if and only if no vector in S is expressible as a linear combination of the other vectors in S.

Example 17
$\mathbf{u} = (1, 1, 0)$, $\mathbf{v} = (-1, 1, 0)$, and $\mathbf{w} = (5, 1, 0)$ are linearly dependent in $\mathbb{R}^3$.
$\mathbf{u} = (1, 1, 0)$, and $\mathbf{v} = (-1, 1, 0)$, and $\mathbf{x} = (1, 1, 1)$ are linearly independent in R^3

Theorem 7
A finite set of vectors that contains the zero vector is linearly dependent.

Proof: DIY

Theorem 8
Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ be a set of vectors in $\mathbb{R}^n$. If $r > n$, then S is linearly dependent.

Proof: The linear system for finding the coefficients has n equations in $r > n$ variables, therefore the system has a solution and the set in linearly dependent.

4 Basis and Dimension

Definition 6
If V is any vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a set of vectors in V, then S is called a basis for V if the following conditions hold

(a) S is linearly independent

(b) S spans V

Example 18
The set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is a basis of $\mathbb{R}^3$.

Theorem 9
If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ is a basis for a vector space V, then every vector of $\mathbf{v} \in V$ can be expressed as

$$\mathbf{v} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_r\mathbf{v_r}$$

in exactly one way.

Example 19
Standard basis for $\mathbb{R}^n$

Example 20
$\{(1,1,1), (1,1,0), (1,0,0)\}$ are a basis for $\mathbb{R}^3$.
Proof:
To prove that the three vectors are a basis, we need to show that they are linearly independent and span $\mathbb{R}^3$.
To show that they are linearly independent we can show that the homogeneous linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$$

only has the trivial solution, which is equivalent to the coefficient matrix being invertible, which is equivalent to the determinant of the coefficient matrix not being 0.

$$det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\right) = 1$$

Therefore the vectors are linearly independent.
To prove that the vectors span $\mathbb{R}^3$ we have to show that the linear system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has a solution for every vector $\mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. This is equivalent to the coefficient matrix being invertible, or the determinant of the coefficient matrix not being zero (this we showed already).
Therefore the vectors span $\mathbb{R}^3$, and therefore $\{(1,1,1), (1,1,0), (1,0,0)\}$ is a basis of $\mathbb{R}^3$.

Example 21
Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ be a set of linearly independent vectors in the vector space V, then S is a basis of $span(S)$.
Because they are linearly independent, and by definition S spans $span(S)$.

Definition 7
A nonzero vector space is called finite dimensional if it contains a finite set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}\}$ that forms a basis. If no such set exists, V is called infinite-dimensional.
The zero vector space is assumed to be finite-dimensional.

Theorem 10
Let V be a finite-dimensional vector space, and let $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be any basis

(a) If a set has more than n vectors then it is linearly dependent.

(b) If set has fewer than n vectors it does not span V.

Theorem 11
All bases of a finite dimensional vector space have the same number of vectors.

Definition 8
The dimension of a finite dimensional vector space V, $dim(V)$, is defined to be the number of vectors in a basis of V.
The zero vector space is defined to have dimension zero.

Example 22
dim($\mathbb{R}^n$)=n, $dim(P_n) = n + 1$, $dim(F(-\infty, \infty)) = \infty$

Example 23
According to a previous result the set of solutions of a linear system form a vector space. Find the dimension of the solution space of the linear system.

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 6x_3 - 2x_4 &= 0 \end{aligned}$$

This system has the same solutions as (2× the first row subtracted from the second row)

$$\begin{aligned} x_1 - 4x_2 + 3x_3 - x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

Then for $s, t, r \in \mathbb{R}$ the solutions are $x_1 = 4s - 3t + r, x_2 = s, x_3 = t, x_4 = r$. Therefore every solution can be written in the form:

$$\mathbf{x} = s(4, 1, 0, 0) + t(-3, 0, 1, 0) + r(1, 0, 0, 1)$$

i.e. every solution of the linear system is a linear combination of vectors in the set $S = \{(4, 1, 0, 0), (9-3, 0, 1, 0), (1, 0, 0, 1)\}$. The solution space $= span(S)$, therefore the dimension is 3.

The following theorems establish connections between the concepts of spanning, linear independence, basis and dimension.

Theorem 12
Let S be a nonempty set of vectors in a vector space V.

(a) If S is linearly independent and if $\mathbf{v}$ is a vector in V, but not in $span(s)$, then the set $S \cup \{v\}$ is linearly independent.

(b) If $\mathbf{v} \in S$ is a linear combination of the other vectors in S, then

$$span(S) = span(S - \{v\})$$

Theorem 13
If V is a n dimensional vector space and S is a set of n linearly independent vectors, then S is a basis of V

Example 24
Vectors $(1, 2)$ and $(-3, 0)$ are linearly independent in $\mathbb{R}^2$, (none is a multiple of the other), therefore the $S = \{(1, 2), (-3, 0)\}$ is a basis of $\mathbb{R}^2$.

Theorem 14
If W is a subspace of a finite-dimensional vector space V, then $dim(W) \leqslant dim(V)$ and if $dim(W) = dim(V)$, then $W = V$.