1 Euclidean Vector Spaces

1.1 Euclidean *n*-space

In this chapter we will generalize the findings from last chapters for a space with n dimensions, called n-space.

Definition 1

If $n \in \mathbb{N}\setminus\{0\}$, then an ordered *n*-tuple is a sequence of *n* numbers in \mathbb{R} : (a_1, a_2, \ldots, a_n) . The set of all ordered *n*-tuples is called *n*-space and is denoted by \mathbb{R}^n .

The elements in \mathbb{R}^n can be perceived as points or vectors, similar to what we have done in 2- and 3-space. (a_1, a_2, a_3) was used to indicate the components of a vector or the coordinates of a point.

Definition 2

Two vectors $\mathbf{u} = (u_1, u_2, \dots, u_n)$ and $\mathbf{v} = (v_1, v_2, \dots, v_n)$ in \mathbb{R}^n are called equal if

$$u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$$

The sum $\mathbf{u} + \mathbf{v}$ is defined by

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \dots u_n + v_n)$$

If $k \in \mathbb{R}$ the scalar multiple of **u** is defined by

 $k\mathbf{u} = (ku_1, ku_2, \dots, ku_n)$

These operations are called the standard operations in \mathbb{R}^n .

Definition 3

The zero vector **0** in \mathbb{R}^n is defined by

 $\mathbf{0} = (0, 0, \dots, 0)$

For $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ the negative of \mathbf{u} is defined by

$$-\mathbf{u} = (-u_1, -u_2, \dots, -u_n)$$

The difference between two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is defined by

$$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$$

Theorem 1 If \mathbf{u}, \mathbf{v} and \mathbf{w} in \mathbb{R}^n and $k, l \in \mathbb{R}$, then

- (a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- (b) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$

- (c) $\mathbf{u} + \mathbf{0} = \mathbf{u}$
- (d) u + (-u) = 0
- (e) $k(l\mathbf{u}) = (kl)\mathbf{u}$
- (f) $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- (g) $(k+l)\mathbf{u} = k\mathbf{u} + l\mathbf{u}$
- (h) $1\mathbf{u}$ = \mathbf{u}

This theorem permits us to manipulate equations without writing them in component form.

Definition 4

If $\mathbf{u} = (u_1, u_2, \dots, u_n), \mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$, then the Euclidean inner product $\mathbf{u} \cdot \mathbf{v}$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Theorem 2

If \mathbf{u}, \mathbf{v} and \mathbf{w} in \mathbb{R}^n and $k \in \mathbb{R}$, then

- (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- (b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- (c) $(k\mathbf{u}) \cdot \mathbf{v} = k(\mathbf{u} \cdot \mathbf{v})$
- (d) $\mathbf{u} \cdot \mathbf{u} \ge 0$.
- (e) $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = 0$.

Proof:

(d) Let $\mathbf{u} \in \mathbb{R}^n$ then $\mathbf{u} \cdot \mathbf{u} = u_1^2 + u_2^2 + \dots u_n^2$, by definition. Since all terms are squares they are greater or equal than zero, and since the sum of numbers greater or equal than zero is also greater or equal than zero we found that $\mathbf{u} \cdot \mathbf{u} \ge 0$.

The total can only be zero if each individual term is zero, that is $u_i^2 = 0$ for all $1 \le i \le n$, but this is equivalent to $u_i = 0$ for $1 \le i \le n$, therefore $\mathbf{u} = \mathbf{0}$, which proves (e).

Definition 5

If $\mathbf{u} \in \mathbb{R}^n$ then the Euclidean norm of \mathbf{u} is defined by

$$||\mathbf{u}|| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$$

The Euclidean distance between two points \mathbf{u} and \mathbf{v} is defined as

$$d(\mathbf{u}, \mathbf{v}) = ||\mathbf{v} - \mathbf{u}||$$

Theorem 3 Cauchy-Schwarz Inequality in \mathbb{R}^n If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then

 $|\mathbf{u} \cdot \mathbf{v}| \leqslant ||\mathbf{u}|| \; ||\mathbf{v}||$

Proof later

Theorem 4 If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $k \in \mathbb{R}$, then:

- (a) $||\mathbf{u}|| \ge 0$
- (b) $||\mathbf{u}|| = 0$ if and only if $\mathbf{u} = \mathbf{0}$
- (c) $||k\mathbf{u}|| = |k| ||\mathbf{u}||$
- (d) $||\mathbf{u} + \mathbf{v}|| \leq ||\mathbf{u}|| + ||\mathbf{v}||$ (triangle inequality)

Proof

(d) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then

$$\begin{aligned} ||\mathbf{u} + \mathbf{v}||^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \\ &= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\ &= ||\mathbf{u}||^2 + 2\mathbf{u} \cdot \mathbf{v} + ||\mathbf{v}||^2 \\ &\leq ||\mathbf{u}||^2 + 2||\mathbf{u} \cdot \mathbf{v}| + ||\mathbf{v}||^2 \quad \text{absolute value} \\ &\leq ||\mathbf{u}||^2 + 2||\mathbf{u}|| \ ||\mathbf{v}|| + ||\mathbf{v}||^2 \quad \text{Cauchy} - \text{Schwarz} \\ &= (||\mathbf{u}|| + ||\mathbf{v}||)^2 \end{aligned}$$

Then the triangle inequality follows by taking the square root on both sides.





- (a) $d(\mathbf{u}, \mathbf{v}) \ge 0$
- (b) $d(\mathbf{u}, \mathbf{v}) = 0$ if and only if $\mathbf{u} = \mathbf{v}$
- (c) $d(\mathbf{u}, \mathbf{v}) = d(\mathbf{v}, \mathbf{u})$

(d) $d(\mathbf{u}, \mathbf{v}) \leq d(\mathbf{u}, \mathbf{w}) + d(\mathbf{w}, \mathbf{v})$ Triangle inequality

Theorem 6

If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, then:

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4} ||\mathbf{u} + \mathbf{v}||^2 - \frac{1}{4} ||\mathbf{u} - \mathbf{v}||^2$$

Proof: For bonus marks?

Definition 6

Two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are called orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.

Motivated by a result in \mathbb{R}^2 and \mathbb{R}^3 we find

Theorem 7 Pythagorean Theorem in \mathbb{R}^n

If **u** and **v** are orthogonal in \mathbb{R}^n , then

$$||\mathbf{u} + \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

Proof: Let \mathbf{u}, \mathbf{v} be orthogonal vectors in \mathbb{R}^n , then $\mathbf{u} \cdot \mathbf{v} = 0$, therefore

$$||\mathbf{u} + \mathbf{v}||^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = ||\mathbf{u}||^2 + 2(\mathbf{u} \cdot \mathbf{v}) + ||\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$$

The dot product and matrix multiplication

Vectors in \mathbb{R}^n can be interpreted as $1 \times n$ or $n \times 1$ matrices. We will identify vectors in \mathbb{R}^n with column vectors in matrix notation, that is $n \times 1$ matrices.

In this case the scalar multiplication and addition in the Euclidean space is equivalent to the scalar multiplication and addition for matrices, respectively.

For the dot product and the matrix multiplication of two vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^n$ the following relationship holds:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \mathbf{v}^T \mathbf{u}$$

and therefore for a $n \times n$ matrix A

$$A\mathbf{u} \cdot \mathbf{v} = \mathbf{v}^T A \mathbf{u} = \mathbf{u} \cdot A^T \mathbf{v}$$

and

$$\mathbf{u} \cdot A\mathbf{v} = \mathbf{v}^T A^T \mathbf{u} = A^T \mathbf{u} \cdot \mathbf{v}$$