1.5 Elementary Matrices and a Method for Finding the Inverse

Definition 1
A $n \times n$ matrix is called an elementary matrix if it can be obtained from I_n by performing a single elementary row operation

Reminder: Elementary row operations:

1. Multiply a row a by $k \in \mathbb{R}$
2. Exchange two rows
3. Add a multiple of one row to another

Theorem 1
If the elementary matrix E results from performing a certain row operation on I_n and A is a $m \times n$ matrix, then EA is the matrix that results when the same row operation is performed on A.

Proof: Has to be done for each elementary row operation.

Example 1

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

E was obtained from I_2 by exchanging the two rows.

$$EA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

EA is the matrix which results from A by exchanging the two rows. (to be expected according to the theorem above.)

Theorem 2
Every elementary matrix is invertible, and the inverse is also an elementary matrix.

Theorem 3
If A is a $n \times n$ matrix then the following statements are equivalent

1. A is invertible
2. $A\mathbf{x} = 0$ has only the trivial solution
3. The reduced echelon form of A is I_n
4. A can be expressed as a product of elementary matrices.

Proof:
Prove the theorem by proving (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a):
(a)$\Rightarrow$(b):
Assume A is invertible and $\mathbf{x_0}$ is a solution of $A\mathbf{x} = 0$, then
$A\mathbf{x_0} = 0$, multiplying both sides with A^{-1} gives
$(A^{-1}A)\mathbf{x_0} = 0 \Leftrightarrow I_n\mathbf{x_0} = 0 \Leftrightarrow \mathbf{x_0} = 0$. Thus $\mathbf{x_0} = 0$ is the only solution of $A\mathbf{x} = 0$.

(b)$\Rightarrow$(c):
Assume that $A\mathbf{x} = 0$ only has the trivial solution then the augmented matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & 0 \end{bmatrix}$$

can be transformed by Gauss-Jordan elimination, so that the corresponding linear system must be (this is the only linear system that only has the trivial solution)

$$\begin{array}{cccccc} x_1 & & & & = & 0 \\ & x_2 & & & = & 0 \\ & & \ddots & & & \vdots \\ & & & x_n & = & 0 \end{array}$$

that is the reduced row echelon form is

$$\begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

This mean the reduced row echelon form of A is I_n.

(c)$\Rightarrow$(d):
Assume the reduced row echelon form of A is I_n. Then I_n is the resulting matrix from Gauss-Jordan Elimination for matrix A (let's say in k steps).
Each of the k transformations in the Gauss-Jordan elimination is equivalent to the multiplication with an elementary matrix.
Let E_i be the elementary matrix corresponding to the i-th transformation in the Gauss-Jordan Elimination for matrix A. Then

$$E_k E_{k-1} \cdots E_2 E_1 A = I_n \tag{1}$$

By the theorem ?? $E_k E_{k-1} \cdots E_2 E_1$ is invertible with inverse $E_1^{-1} E_2^{-1} \cdots E_k^{-1}$.
Multiplying above equation on both sides by the inverse from the left we get

$$(E_1^{-1} E_2^{-1} \cdots E_k^{-1}) E_k E_{k-1} \cdots E_2 E_1 A = (E_1^{-1} E_2^{-1} \cdots E_k^{-1}) I_n \Leftrightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

Since the inverse of elementary matrices are also elementary matrices, we found that A can be expressed as a product of elementary matrices.

(d)⇒(a):
If A can be expressed as a product of elementary matrices, then A can be expressed as a product of invertible matrices, therefore is invertible (theorem ??).
From this theorem we obtain a method to the inverse of a matrix A.
Multiplying equation 5 from the right by A^{-1} yields

$$E_k E_{k-1} \cdots E_2 E_1 I_n = A^{-1}$$

Therefore, when applying the elementary row operations, that transform A to I_n, to the matrix I_n we obtain A^{-1}.
The following example illustrates how this result can be used to find A^{-1}.

Example 2

1.
$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

Reduce A through elementary row operations to the identity matrix, while applying the same operations to I_3, resulting in A^{-1}.
To do so we set up a matrix

$$[A|I] \overset{\text{row operations}}{\longrightarrow} [I|A^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$-(r1)+r2, -2(r1)+r3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & -1 & 1 & 0 \\ 0 & -3 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$-1/2(r2) \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & -3 & 2 & -2 & 0 & 1 \end{array}\right]$$

$$3(r2)+r3 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1/2 & -1/2 & -3/2 & 1 \end{array}\right]$$

$$2(r3) \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1 & -3 & 2 \end{array}\right]$$

$$1/2(r3)+r2, -(r3)+r1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 2 & 3 & -2 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & -3 & 2 \end{array}\right]$$

$$-3(r2)+r1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 9 & -5 \\ 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & -3 & 2 \end{array}\right]$$

The inverse of A is

$$A^{-1} = \begin{bmatrix} 2 & 9 & -5 \\ 0 & -2 & 1 \\ -1 & -3 & 2 \end{bmatrix}$$

2. Consider the matrix

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$$

Setup $[A|I]$ and transform

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 2 & -6 & 0 & 1 \end{array}\right]$$

$$-2(r1)+r2 \rightarrow \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

The last row is a zero row, indicating that A can not be transformed into I_2 by elementary row operation, according to theorem ??, A must be singular, i.e. A is not invertible and has no inverse.

1.6 Results of Linear Systems and Invertibility

Theorem 4
Every system of linear equations has no solution, exactly one solution, or infinitely many solutions.

Proof:
If $A\mathbf{x} = \mathbf{b}$ is a linear system, then it can have no solution, this we saw in examples.
So all we need to show is that, if the linear system has two different solutions then it has to have infinitely many solutions.
Assume $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_1} \neq \mathbf{x_2}$ are two different solutions of $A\mathbf{x} = \mathbf{b}$,
therefore $A\mathbf{x_1} = \mathbf{b}$ and $A\mathbf{x_2} = \mathbf{b}$.
Let $k \in \mathbb{R}$, then I claim $\mathbf{x_1} + k(\mathbf{x_1} - \mathbf{x_2})$ is also a solution of $A\mathbf{x} = \mathbf{b}$. To show that this is true that $A(\mathbf{x_1} + k(\mathbf{x_1} - \mathbf{x_2})) = \mathbf{b}$.
Calculate

$$\begin{aligned} A(\mathbf{x_1} + k(\mathbf{x_1} - \mathbf{x_2})) &= A\mathbf{x_1} + kA(\mathbf{x_1} - \mathbf{x_2}) \\ &= \mathbf{b} + kA\mathbf{x_1} - kA\mathbf{x_2} \\ &= \mathbf{b} + k\mathbf{b} - k\mathbf{b} \\ &= \mathbf{b} \end{aligned}$$

Theorem 5
If A is an invertible $n \times n$ matrix, then for each $n \times 1$ vector $\mathbf{b}$, the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Proof:
First show that $\mathbf{x} = A^{-1}\mathbf{b}$ is a solution Calculate

$$\begin{aligned} A\mathbf{x} &= AA^{-1}\mathbf{b} \\ &= I\mathbf{b} = \mathbf{b} \end{aligned}$$

Now we have to show that if there is any solution it must be $A^{-1}\mathbf{b}$.
Assume $\mathbf{x_0}$ is a solution, then

$$\begin{aligned} & A\mathbf{x_0} &= \mathbf{b} \\ \Leftrightarrow\ & A^{-1}A\mathbf{x_0} &= A^{-1}\mathbf{b} \\ \Leftrightarrow\ & \mathbf{x_0} &= A^{-1}\mathbf{b} \end{aligned}$$

We proved, that if $\mathbf{x_0}$ is a solution then it must be $A^{-1}\mathbf{x_0}$, which concludes the proof.

Example 3
Let us solve

$$A = \begin{array}{c} 1x_1 + 3x_2 + x_3 = 4 \\ 1x_1 + 1x_2 + 2x_3 = 2 \\ 2x_1 + 3x_2 + 4x_3 = 1 \end{array}$$

Then the coefficient matrix is

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$

with (see example above)

$$A^{-1} = \begin{bmatrix} 2 & 9 & -5 \\ 0 & -2 & 1 \\ -1 & -3 & 2 \end{bmatrix}$$

Then the solution of above linear system is

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 9 & -5 \\ 0 & -2 & 1 \\ -1 & -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2(4) + 9(2) - 5(1) \\ 0(4) - 2(2) + 1(1) \\ -1(4) - 3(2) + 2(1) \end{bmatrix} = \begin{bmatrix} 21 \\ -3 \\ -8 \end{bmatrix}$$

Theorem 6
Let A be a square matrix.

(a) If B is a square matrix with $BA = I$, then $B = A^{-1}$

(b) If B is a square matrix with $AB = I$, then $B = A^{-1}$

Proof: Assume $BA = I$.
First show, that A is invertible. According to the equivalence theorem, we only have to show that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, then A must be invertible.

$$\begin{aligned} A\mathbf{x} &= \mathbf{0} \\ \Leftrightarrow \quad BA\mathbf{x} &= B\mathbf{0} \\ \Leftrightarrow \quad I\mathbf{x} &= \mathbf{0} \\ \Leftrightarrow \quad \mathbf{x} &= \mathbf{0} \end{aligned}$$

Therefor if $\mathbf{x}$ is a solution then $\mathbf{x} = \mathbf{0}$, thus the linear system has only the trivial solution, and A must be invertible.
To show that $B = A^{-1}$

$$\begin{aligned} BA &= I \\ \Leftrightarrow \quad BAA^{-1} &= IA^{-1} \\ \Leftrightarrow \quad B &= A^{-1} \end{aligned}$$

So $B = A^{-1}$ which concludes the proof for part (a), try (b) by yourself.
Extend the equivalence theorem

Theorem 7

(a) A is invertible

(b) For every $n \times 1$ vector $\mathbf{b}$ the linear system $A\mathbf{x} = \mathbf{b}$ is consistent.

(c) For every $n \times 1$ vector $\mathbf{b}$ the linear system $A\mathbf{x} = \mathbf{b}$ has exactly one solution.

Proof:
Show (a)$\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a)
(a)$\Rightarrow$ (c): This we proved above.
(c)$\Rightarrow$ (b): If the linear system has exactly one solution for all $n \times 1$ vector $\mathbf{b}$, then it has at least one solution. Thus (b) holds.

(b) $\Rightarrow$ (a): Assume for every $n \times 1$ vector $\mathbf{b}$ the linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution. Then let $\mathbf{x_1}$ be a solution of

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and $\mathbf{x_2}$ be a solution of

$$A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

choose $\mathbf{x_3}, \ldots, \mathbf{x_n}$ similar.
Then let

$$C = [\mathbf{x_1} | \mathbf{x_2} | \cdots | \mathbf{x_n}]$$

Then

$$AC = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \cdots \middle| \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right] = I_n$$

According to theorem above then A is invertible, thus (a) holds.

Example 4
One common problem that arises is the following:
Given that we have a matrix A, what are the vectors $\mathbf{b}$, so that $A\mathbf{x} = \mathbf{b}$ has a solution.
Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

then we are asking for which values of b_1, b_2 does the linear system

$$\begin{aligned} x_1 + 1x_2 &= b_1 \\ x_1 + -2x_2 &= b_2 \end{aligned}$$

have a solution?
Use Gaussian Elimination to give the answer. The augmented matrix is

$$\begin{bmatrix} 1 & 1 & b_1 \\ 1 & -2 & b_2 \end{bmatrix} \overset{r2-r1}{\longrightarrow} \begin{bmatrix} 1 & 1 & b_1 \\ 0 & -3 & b_2 - b_1 \end{bmatrix} \overset{-1/3(r2)}{\longrightarrow}$$

$$\begin{bmatrix} 1 & 1 & b_1 \\ 0 & 1 & -(1/3)b_2 + (1/3)b_1) \end{bmatrix} \overset{r1-r2}{\longrightarrow} \begin{bmatrix} 1 & 0 & (2/3)b_1 + (1/3)b_2 \\ 0 & 1 & -(1/3)b_2 + (1/3)b_1) \end{bmatrix}$$

From this we conclude, that for any b_1, b_2 the linear system has a solution, which is $x_1 = (2/3)b_1 + (1/3)b_2, x_2 = -(1/3)b_2 + (1/3)b_1)$

1.7 Special Matrices

Definition 2
A square matrix A is called diagonal, iff all entries NOT on the main diagonal are 0, i.e. for all $i \neq j : a_{ij} = 0$.

Example 5

1.
$$A = \begin{bmatrix} 51 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

is diagonal. The inverse is
$$A^{-1} = \begin{bmatrix} 1/51 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/4 \end{bmatrix}$$

The inverse of a diagonal matrix is also diagonal, with entries $(A^{-1})_{ii} = 1/(A)_{ii}$ on the diagonal.

$$A^k = \begin{bmatrix} 51^2 & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & (-4)^k \end{bmatrix}$$

The power of a diagonal matrix is also a diagonal matrix, with entries $(A^k)_{ii} = ((A)_{ii})^k$ on the diagonal.

Definition 3

A square matrix A is called an upper triangular matrix, if all entries below the main diagonal are zero, i.e. for $i > j : a_{ij} = 0$.

A square matrix A is called a lower triangular matrix, if all entries above the main diagonal are zero, i.e. for $i < j : a_{ij} = 0$.

Theorem 8

(a) The transpose of a lower triangular matrix is upper triangular, and the transpose of an upper triangular matrix is lower triangular.

(b) The product of lower(upper) triangular matrices is lower(upper) triangular.

(c) A triangular matrix is invertible, iff all diagonal entries are nonzero.

(d) The inverse of an invertible lower(upper) triangular matrix is lower(upper) triangular.

Example 6
Consider the following matrices

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}$$

then A is invertible, because all diagonal entries are not zero, but B is not invertible $b_{22} = 0$.
Calculate

$$AB = \begin{bmatrix} 1 & 2 & 8 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

which is like predicted by the last theorem upper triangular.

Definition 4
A square matrix is called symmetric iff $A = A^T$, or $a_{ij} = a_{ji}$.

Example 7

$$A = \begin{bmatrix} 1 & 5 & 3 \\ 5 & 2 & 2 \\ 3 & 2 & -1 \end{bmatrix}$$

is symmetric.

Theorem 9
If A and B are symmetric matrices of the same size, and $k \in \mathbb{R}$, then:

(a) A^T is symmetric.

(b) $A + B$ is symmetric

(c) kA is symmetric.

Proof:

(a) Since $A^T = A$ and A is symmetric so must be A^T.

(b) $(A + B)_{ji} = (A)_{ji} + (B)_{ji} = (A)_{ij} + (B)_{ij} = (A + B)_{ij}$, so $A + B$ is symmetric.

(c) similar to (b).

Remark: Usually AB is not symmetric, even if A and B are symmetric. But if the matrices commute, i.e. $AB = BA$, then AB is a symmetric matrix.

Theorem 10
If A is an invertible symmetric matrix, then A^{-1} is also symmetric.

Proof:
Assume A is invertible and $A = A^T$, then $(A^{-1})^T \overset{\text{theorem before}}{=} (A^T)^{-1} = A^{-1}$, so A^{-1} is symmetric.

Remark: In many situations for a $m \times n$ matrix A the matrices A^TA and AA^T are of interest. A^TA is a $n \times n$ matrix and AA^T is a $m \times m$ matrix. Both are symmetric, because

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

similar for AA^T

Example 8
Let

$$A = \begin{bmatrix} 1 & 5 & -3 \\ 5 & -2 & 2 \end{bmatrix}$$

then

$$AA^T = \begin{bmatrix} 1 & 5 & -3 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 5 & -2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 35 & -11 \\ -11 & 33 \end{bmatrix}$$

and

$$A^TA = \begin{bmatrix} 1 & 5 \\ 5 & -2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 5 & -3 \\ 5 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 26 & -5 & 7 \\ -5 & 29 & -19 \\ 7 & -19 & 13 \end{bmatrix}$$

Theorem 11
If A is invertible then A^TA and AA^T are also invertible

Proof: Since A is invertible, then A^T is also invertible, because a product of invertible matrices is invertible we conclude that AA^T and A^TA must also be invertible.