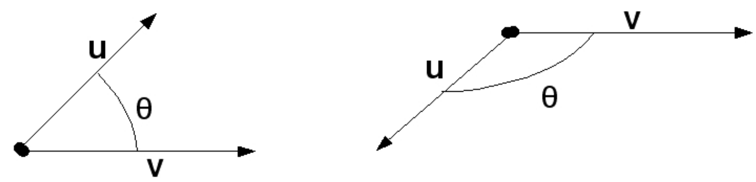
0.1 Dot Product and Projections

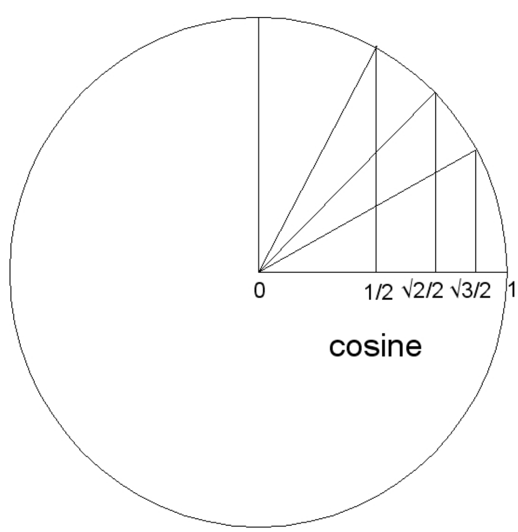
Definition 1
Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors, assume that both vector have been positioned to have the same initial point, then the angle between $\mathbf{u}$ and $\mathbf{v}$ is the angle θ determined by $\mathbf{u}$ and $\mathbf{v}$ that satisfies $0 \leqslant \theta \leqslant \pi$.



Definition 2
Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors and θ the angle between $\mathbf{u}$ and $\mathbf{v}$, then the dot product or Euclidean inner product, $\mathbf{u} \cdot \mathbf{v}$, is defined by

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} ||\mathbf{u}|| \; ||\mathbf{v}|| cos(\theta) & \text{if } \mathbf{u} \neq \mathbf{0} \text{ and } \mathbf{v} \neq \mathbf{0} \\ 0 & \text{if } \mathbf{u} = \mathbf{0} \text{ or } \mathbf{v} = \mathbf{0} \end{cases}$$

θ out of 360^o	2π	$cos(\theta)$	360^o	2π	$cos(\theta)$
0^o	0	$\sqrt{4}/2 = 1$	120^o	$2\pi/3$	$-\sqrt{1}/2 = -1/2$
30^o	$\pi/6$	$\sqrt{3}/2$	135^o	$3\pi/4$	$-\sqrt{2}/2$
45^o	$\pi/4$	$\sqrt{2}/2$	150^o	$5\pi/6$	$-\sqrt{3}/2$
60^o	$\pi/3$	$\sqrt{1}/2 = 1/2$	180^o	π	$-\sqrt{4}/2 = -1$
90^o	$\pi/2$	$\sqrt{0}/2 = 0$			



Example 1
The dot product of vector $\mathbf{u} = (0, 1)$ and $\mathbf{v} = (1, 1)$ is

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \; ||\mathbf{v}|| \; cos(45^o) = \sqrt{1} \; \sqrt{2}\frac{\sqrt{2}}{2} = 1$$

Component form of dot product
From the cosine law, we get

$$||\overrightarrow{PQ}|| = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2||\mathbf{u}||\ ||\mathbf{v}||cos(\theta)$$

Since $\overrightarrow{PQ} = \mathbf{v} - \mathbf{u}$, we get

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= ||\mathbf{u}||\ ||\mathbf{v}||cos(\theta) \\
&= \tfrac{1}{2}(||\mathbf{u}||^2 + ||\mathbf{v}||^2 + ||\mathbf{v}-\mathbf{u}||^2) \\
&= \tfrac{1}{2}(u_1^2 + u_2^2 + v_1^2 + v_2^2 + (v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2) \\
&= \tfrac{1}{2}(2u_1v_1 + 2u_2v_2 + 2u_3v_3) \\
&= u_1v_1 + u_2v_2 + u_3v_3
\end{aligned}$$

(Remember, how this relates to matrix multiplication: The dot product is the same as $\mathbf{u}\mathbf{v}^T$ (cool?))
We get

$$cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||}$$

Example 2
Example find angle between two vectors
Consider vectors $\mathbf{u} = (2, 0, 2)$ and $\mathbf{v} = (1, 1, 1)$, then for θ we get

$$cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\ ||\mathbf{v}||} = \frac{(2)(1) + (0)(1) + (2)(1)}{\sqrt{8}\ \sqrt{3}} = \frac{4}{2\ \sqrt{6}} = 0.816$$

i.e. $\theta \approx 35.26^o$

Theorem 1
Let $\mathbf{u}$ and $\mathbf{v}$ be vector in 2- or 3-space

(a) $\mathbf{v}\cdot\mathbf{v} = ||v||^2$

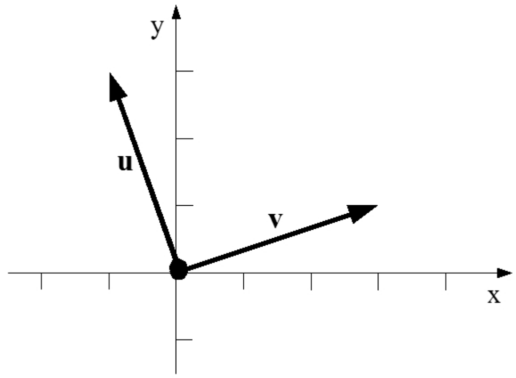
(b) If $\mathbf{u}$ and $\mathbf{v}$ are non zero and θ is the angle between the two vectors then

θ is acute if and only if $\mathbf{u}\cdot\mathbf{v} > 0$

θ is obtuse if and only if $\mathbf{u}\cdot\mathbf{v} < 0$

$\theta = \pi/2$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$

Example 3
Let $\mathbf{u} = (-1, 3)$ and $\mathbf{v} = (3, 1)$, then $\mathbf{u}\cdot\mathbf{v} = (-1)3 + (3)(1) = 0$, the two vectors are orthogonal.



Two vector that are perpendicular are also called orthogonal

Theorem 2
Two nonzero vector $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$

Example 4
In 2-space the nonzero vector $\mathbf{n} = (a, b)$ is orthogonal to the line given by $ax + by + c = 0$.
To prove this claim, let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be distinct points on the line, therefore

$$ax_1 + by_1 + c = 0 \Leftrightarrow ax_1 + by_1 = -c$$
$$ax_2 + by_2 + c = 0 \Leftrightarrow ax_2 + by_2 = -c$$

Since $\overrightarrow{P_1P_2}(x_2 - x_1, y_2 - y_1)$ is on the line we need to prove that $\mathbf{n} \cdot \overrightarrow{P_1P_2} = 0$.

$$\begin{aligned}
\mathbf{n} \cdot \overrightarrow{P_1P_2} &= (a, b) \cdot (x_2 - x_1, y_2 - y_1) \\
&= a(x_2 - x_1) + b(y_2 - y_1) \\
&= ax_2 + by_2 - (ax_1 + by_1) \quad \text{(see above)} \\
&= -c - (-c) \\
&= 0
\end{aligned}$$

therefore the vectors $\mathbf{n}$ and $\overrightarrow{P_1P_2}$ are orthogonal, therefore $\mathbf{n}$ and the line given by $ax + by + c = 0$ are orthogonal.

Theorem 3
If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, and $k \in \mathbb{R}$, then

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

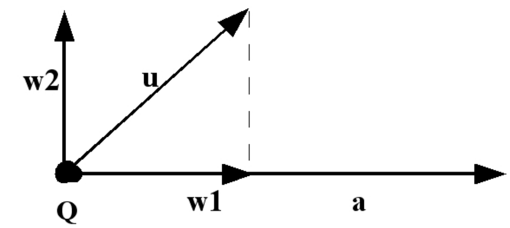
(b) $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

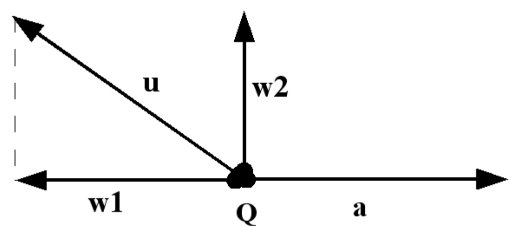
(c) $k(\mathbf{u} \cdot \mathbf{v}) = (k\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (k\mathbf{v})$

(d) If $\mathbf{v} \neq 0$ then $\mathbf{v} \cdot \mathbf{v} > 0$, and if $\mathbf{v} = 0$, then $\mathbf{v} \cdot \mathbf{v} = 0$

Proof: Do it yourself.
Orthogonal Projection
Many problems can be solved analyzing a vector $\mathbf{u}$ into two terms, one parallel to another vector $\mathbf{a}$ and the second being orthogonal to $\mathbf{a}$.
If $\mathbf{u}$ and $\mathbf{a}$ have the same initial points Q, we can decompose $\mathbf{u}$ as follows:

1. Drop an orthogonal from the terminal point of $\mathbf{u}$ on the line through $\mathbf{a}$, construct vector $\mathbf{w_1}$ from Q to the point on the line through $\mathbf{a}$.
2. Find $\mathbf{w_2} = \mathbf{u} - \mathbf{w_1}$





The vector $\mathbf{w_1}$ is called the orthogonal projection of $\mathbf{u}$ on $\mathbf{a}$, or the vector component of $\mathbf{u}$ along $\mathbf{a}$. Denotation

$$proj_{\mathbf{a}}\mathbf{u}$$

and $\mathbf{w_2}$ is called the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.
It is always true that

$$\mathbf{u} = \mathbf{w_1} + \mathbf{w_2}$$

Theorem 4
If $\mathbf{u}$ and $\mathbf{a}$ are vectors, then the projection of $\mathbf{u}$ on $\mathbf{a}$ is

$$proj_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2}\ \mathbf{a}$$

and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - proj_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u} \cdot \mathbf{a}}{||\mathbf{a}||^2}\ \mathbf{a}$$

Proof: (see text book pg 140)

Example 5
Let $\mathbf{u} = (1, 2, -1)$ and $\mathbf{a} = (-2, 3, 0)$, then the projection of $\mathbf{u}$ on $\mathbf{a}$ is

$$proj_{\mathbf{a}}\mathbf{u} = \frac{1(-2) + 2(3) + (-1)0}{((-2)^2 + 3^2 + 0^2)}\ (-2, 3, 0) = \frac{4}{13}\ (-2, 3, 0) = (\frac{-8}{13}, \frac{12}{13}, 0)$$

and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - proj_{\mathbf{a}}\mathbf{u} = (1, 2, -1) - (\frac{-8}{13}, \frac{12}{13}, 0) = (\frac{21}{13}, \frac{14}{13}, -1)$$

Theorem 5
For vector **u** and **a**

$$||proj_{\mathbf{a}}\mathbf{u}|| = ||u||\ |cos(\theta)|$$

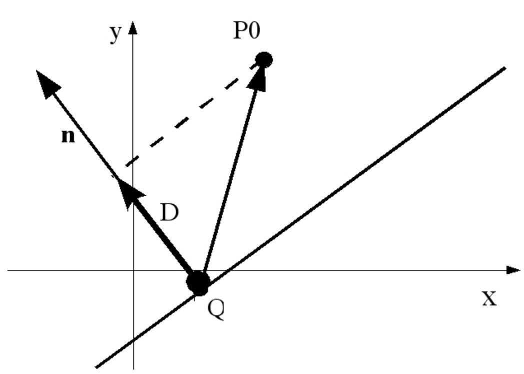
if θ is the angle between **u** and **a**.

Proof:
Let **u** and **a** vectors, and θ is the angle between **u** and **a**, then

$$\begin{aligned}
||proj_{\mathbf{a}}\mathbf{u}|| &= \left|\left|\frac{\mathbf{u}\cdot\mathbf{a}}{||\mathbf{a}||^2}\ \mathbf{a}\right|\right| \\
&= \left|\frac{\mathbf{u}\cdot\mathbf{a}}{||\mathbf{a}||^2}\right|\ ||\mathbf{a}|| \\
&= \frac{|\mathbf{u}\cdot\mathbf{a}|}{||\mathbf{a}||^2}\ ||\mathbf{a}|| \\
&= \frac{|\mathbf{u}\cdot\mathbf{a}|}{||\mathbf{a}||} \\
&= \frac{|\ ||\mathbf{u}||\ ||\mathbf{a}||\ cos(\theta)\ |}{||\mathbf{a}||} \\
&= \frac{||\mathbf{u}||\ ||\mathbf{a}||\ |cos(\theta)|}{||\mathbf{a}||} \\
&= ||\mathbf{u}||\ |cos(\theta)\ |
\end{aligned}$$

Example 6
To find the distance D between a point $P_0(x_0, y_0)$ and a line given through $ax + by + c = 0$, choose any point $Q(x_1, y_1)$ on the line, that is any (x_1, y_1) with

$$ax_1 + by_1 + c = 0 \quad \text{or} \quad c = -ax_1 - by_1 \qquad (*)$$



We know that $\mathbf{n} = (a, b)$ is orthogonal to the line, then

$$D = ||proj_{\mathbf{n}}\overrightarrow{QP_0}||$$

(see diagram below)
Therefore we get (see proof of theorem)

$$\begin{aligned} D &= \frac{|\overrightarrow{QP_0} \cdot \mathbf{n}|}{||\mathbf{n}||} \\ &= \frac{|(x_0 - x_1, y_0 - y_1) \cdot (a, b)|}{\sqrt{a^2 + b^2}} \\ &= \frac{|a(x_0 - x_1) + b(y_0 - y_1)|}{\sqrt{a^2 + b^2}} \end{aligned}$$

Substituting $(*)$ into the nominator, we get

$$D = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

Example 7
The distance between the point (1,2) and the line given through $x + 2y - 1 = 0$ is

$$D = \frac{|(1)(1) + 2(2) + (-1)|}{\sqrt{1 + 4}} = \frac{4}{\sqrt{5}}$$