1 Determinants

The determinant of a square matrix is a value in $\mathbb{R}$ assigned to the matrix, it characterizes matrices which are invertible (det$\neq 0$) and is related to the volume of a parallelpiped described by the matrix. The determinant can also be used to find the solutions of linear systems and is therefore a helpful tool in matrix algebra.
The determinant will be defined recursively, i.e. we will first define the determinant for a 2×2 matrix, then we will define the determinant of a $n \times n$ matrix based on determinants of $(n-1)\times(n-1)$ matrices. Applying these rules recursively will lead to the determinant.

Definition 1

If A is a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the determinant of A is defined by $det(A) = |A| = ad - cb$.

(Cofactor Expansion along the first row) If A is a square matrix of size n the

$$det(A) = |A| = \sum_{j=1}^{n} a_{1j}C_{1j}$$

where the cofactor of the entry a_{ij} is C_{ij} defined as

$$C_{ij} = (-1)^{i+j}M_{ij}$$

where the minor of entry a_{ij} is M_{ij}, the determinant of the submatrix that remains after the ith row and jth column are deleted from A

Example 1

(a) Let

$$A = \begin{bmatrix} 1 & 5 \\ 5 & -2 \end{bmatrix}$$

then

$$det(A) = \begin{vmatrix} 1 & 5 \\ 5 & -2 \end{vmatrix} = 1(-2) - 5(5) = -27$$

This was easy because A is a 2×2 matrix

(b) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -1 \\ -3 & 3 & 1 \end{bmatrix}$$

Then the minor (and cofactor) of a_{11} is (delete row 1 and column 1)

$$M_{11} = \begin{vmatrix} -2 & -1 \\ 3 & 1 \end{vmatrix} = -2(1) - 3(-1) = 1, \text{ so } C_{11} = (-1)^{1+1}M_{11} = 1$$

The minor (and cofactor) of a_{12} is (delete row 1 and column 2)

$$M_{12} = \begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix} = 1(1) - (-3)(-1) = -2, \text{ so } C_{12} = (-1)^{1+2} M_{12} = (-1)(-2) = 2$$

The minor (and cofactor) of a_{13} is (delete row 1 and column 3)

$$M_{13} = \begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} = 1(3) - (-3)(-2) = -3, \text{ so } C_{13} = (-1)^{1+3} M_{13} = -3$$

With these

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = 1(1) + 2(2) + 3(-3) = 1 + 4 - 9 = -4$$

The definition is based on the cofactor expansion along the first row. One can prove that it is possible to expand along any row or column

Theorem 1

(a) Expansion along row i

$$det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots a_{in}C_{in}$$

(b) Expansion along column j

$$det(A) = \sum_{i=1}^{n} a_{ij}C_{ij} = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots a_{nj}C_{nj}$$

no proof.

Example 2
The last theorem allows to make smart choice, when calculating a determinant. Let Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & -2 & -1 & 0 \\ -3 & 3 & 1 & 0 \\ 0 & 0 & 6 & 0 \end{bmatrix}$$

For finding the determinant expansion alon the 4th column looks really easy because

$$det(A) = 5(-1)^{1+4} \begin{vmatrix} 1 & -2 & -1 \\ -3 & 3 & 1 \\ 0 & 0 & 6 \end{vmatrix}$$

all other entries are zero and do not provide any more terms. To find the minor M_{14} it is easiest to expand along the third row

$$det(A) = (-5)(6)(-1)^{3+3} \begin{vmatrix} 1 & -2 \\ -3 & 3 \end{vmatrix} = (-5)6(1(3) - (-3)(-2)) = (-5)6(-3) = 90$$

Definition 2

If A is a square matrix of size n and C_{ij} is the cofactor of a_{ij}, then the matrix

$$\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$$

is called the matrix of cofactors from A.

The adjoint of A, $adj(A)$, is defined to be the transpose of the matrix of cofactors for A.

Example 3

Use A from example 1(b):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -1 \\ -3 & 3 & 1 \end{bmatrix}$$

then the cofactors of A are

$$\begin{array}{lll} C_{11} = 1 & C_{12} = 2 & C_{13} = -3 \\ C_{21} = 7 & C_{22} = 10 & C_{23} = -9 \\ C_{31} = 4 & C_{32} = 4 & C_{33} = -4 \end{array}$$

and

$$adj(A) = \begin{bmatrix} 1 & 7 & 4 \\ 2 & 10 & 4 \\ -3 & -9 & -4 \end{bmatrix}$$

Theorem 2

If A is an invertible matrix then

$$A^{-1} = \frac{1}{det(A)}\, adj(A)$$

Example 4

Use matrix A from example 3, then

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & -1 \\ -3 & 3 & 1 \end{bmatrix}, \; adj(A) = \begin{bmatrix} 1 & 7 & 4 \\ 2 & 10 & 4 \\ -3 & -9 & -4 \end{bmatrix}$$

and from ex. 1(b), we know $det(A) = -4$. The last theorem now let us know that

$$A^{-1} = \frac{-1}{4} \begin{bmatrix} 1 & 7 & 4 \\ 2 & 10 & 4 \\ -3 & -9 & -4 \end{bmatrix}$$

Theorem 3

If A is an $n \times n$ matrix (upper triangular, lower triangular, or diagonal), then $det(A)$ is the product of the entries on the diagonal of the matrix, that is

$$det(A) = a_{11}a_{22}\cdots a_{nn}$$

Proof:
Expansion along row 1 to row (column) n successively for lower (upper) triangular matrices shows the result.

Example 5

(a) Let
$$A = \begin{bmatrix} 1 & 7 & 4 \\ 0 & 10 & 4 \\ 0 & 0 & -4 \end{bmatrix}$$
then according to the theorem $det(A) = 1(10)(-4) = -40$. That was easy.

(b) Let
$$B = \begin{bmatrix} 1 & 7 & 4 \\ 0 & 10 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$
Then $det(B) = 0$.

In order to take advantage of this property we will see how we can use elementary row operations to transform a matrix into a upper triangular matrix to find the determinant.

1.1 Row Reductions to Find Determinants

Theorem 4
Let A be a square matrix. If A has a row of zeros or a column of zeros, then $det(A) = 0$.

Proof: Do cofactor expansion along the row(column) that is all zero and you find that the determinant has to be equal to zero.

Theorem 5
Let A be a square matrix then $det(A) = det(A^T)$.

Proof: The determinant of A can be found by expansion along row 1 this is equal to the cofactor expansion along column 1 of the transposed matrix..

Theorem 6
Let A be a square matrix of size n.

(a) If B is the matrix that results when a single row(column) of A is multiplied by $k \in \mathbb{R}$, then $det(B) = kdet(A)$.

(b) If B is the matrix that results when two rows or two columns of A are interchanged, then $det(B) = -det(A)$.

(c) If B is the matrix that results when a multiple of one row(column) of A is added to another row(column), then $det(B) = det(A)$

From this theorem we see the effect of elementary row(column) operations on the determinant, this will help to find the determinant because now we can transform A into a upper or lower triangular form, and then easily find the determinant.

Example 6
Find $det(A)$ for

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & -2 & -1 \\ -3 & 3 & 1 \end{bmatrix}$$

Transform A into row echelon form

$$det(A) = \begin{vmatrix} 0 & 2 & 3 \\ 1 & -2 & -1 \\ -3 & 3 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & -1 \\ 0 & 2 & 3 \\ -3 & 3 & 1 \end{vmatrix} = - \begin{vmatrix} 1 & -2 & -1 \\ 0 & 2 & 3 \\ 0 & -3 & -2 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 3/2 \\ 0 & -3 & -2 \end{vmatrix} = -2 \begin{vmatrix} 1 & -2 & -1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 5/2 \end{vmatrix} = -2(1)(1)(5/2) = -5$$

Because of the theorem above we get

Theorem 7

(a) If E results from multiplying a row of I_n by $k \in \mathbb{R}$, then $det(E) = k$.

(b) If E results from interchanging two rows of I_n , then $det(E) = -1$.

(c) If E results from adding a multiple of one row of I_n to another, then $det(E) = 1$.

Example 7

(a)

$$\begin{vmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 5, \quad \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1, \quad \begin{vmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Theorem 8
If A is a square matrix with two proportional columns(rows), then $det(A) = 0$.

Proof:
Using the elementary row operation of subtracting the multiple of one row (column) to another row (column) will transform A into a matrix with a row (column) of zeros, and this matrix for that reason has a determinant of zero.
Since this elementary row operation does not change the determinant, the determinant of A must be 0.
The next theorem shows how determinants can be used to find solutions of linear systems.

Theorem 9
Cramer's Rule
If $A\mathbf{x} = \mathbf{b}$ is a linear system in n unknowns such that $det(A) \neq 0$, then the system has a unique solution, which is

$$x_1 = \frac{det(A_1)}{det(A)},\ x_2 = \frac{det(A_2)}{det(A)},\ \ldots,\ x_n = \frac{det(A_n)}{det(A)}$$

where A_j is the matrix obtained by replacing the entries in column j of A by the entries of the solution vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Proof:
If $det(A) \neq 0$ then A is invertible and $\mathbf{x} = A^{-1}\mathbf{b}$ is the unique solution of the linear system Using Theorem **??** we get

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{det(A)}\ adj(A)\mathbf{b} = \frac{1}{det(A)} \begin{bmatrix} C_{11} & C_{21} & \ldots & C_{n1} \\ C_{12} & C_{22} & \ldots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \ldots & C_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

resulting in

$$\mathbf{x} = \frac{1}{det(A)} \begin{bmatrix} b_1C_{11} + b_2C_{21} + \ldots + b_nC_{n1} \\ b_1C_{12} + b_2C_{22} + \ldots + b_nC_{n2} \\ \vdots \\ b_1C_{1n} + b_2C_{2n} + \ldots + b_nC_{nn} \end{bmatrix}$$

so

$$x_j = \frac{b_1C_{1j} + b_2C_{2j} + \ldots + b_nC_{nj}}{\det(A)}$$

If

$$A_j = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j-1} & b_1 & a_{1j+1} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j-1} & b_2 & a_{1j+1} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nj-1} & b_n & a_{1j+1} & \ldots & a_{nn} \end{bmatrix}$$

The cofactors of this matrix are equal to the cofactors of A for all entries in column j. I.e. when calculating the determinant of A_j by expanding along column j, one gets

$$det(A_j) = b_1C_{1j} + b_2C_{2j} + \ldots + b_nC_{nj}$$

substituting this into the equation above we find

$$x_j = \frac{det(A_j)}{det(A)}$$

which concludes the proof.

Example 8
Cramer's Rule Assume the following is the augmented matrix of a linear system

$$\begin{bmatrix} 5 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Using Cramer's Rule we can now give the solution right away:

$$x_1 = \frac{det(A_1)}{det(A)} = \frac{2(1)-3(1)}{5(1)-1(1)} = \frac{-1}{4},\ x_2 = \frac{det(A_2)}{det(A)} = \frac{5(3)-1(2)}{5(1)-1(1)} = \frac{13}{4}$$

1.2 Properties of The Determinant

Theorem 10
Let $k \in \mathbb{R}$, and A be a square matrix then

$$det(kA) = k^n det(A)$$

Example 9
In general $det(A+B) \neq det(A) + det(B)$ Let

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 4 & -1 \end{bmatrix}$$

then

$$A+B = \begin{bmatrix} 6 & 4 \\ 5 & 2 \end{bmatrix}$$

and $det(A) = 13,\ det(B) = -9,\ det(A+B) = -8,$ so $det(A) + det(B) \neq det(A+B)$

There following theorem shows, when determinant can be added

Theorem 11
Let A, B, C square matrices of the same size, which only differ in a single row, say the rth row. Assume that the rth row of C is obtained by adding the corresponding entries in the rth row of A and B, then

$$det(C) = det(A) + det(B)$$

Example 10
Illustrate the previous theorem with this example. Let

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix},\ B = \begin{bmatrix} 5 & 2 \\ 4 & -1 \end{bmatrix}, C = \begin{bmatrix} 5 & 2 \\ 1+4 & 3-1 \end{bmatrix}$$

then $det(A) = 13$, $det(B) = -13$, $det(C) = 0$, in this example truly as predicted by the theorem $det(C) = det(A) + det(B)$.

Theorem 12
A square matrix A is invertible if and only if $det(A) \neq 0$

Theorem 13
If A and B are square matrices of the same size then

$$det(AB) = det(A)det(B)$$

Theorem 14
If A is an invertible square matrix then

$$det(A^{-1}) = \frac{1}{det(A)}$$

Proof:
Since $A^{-1}A = I$, therefore $det(A^{-1}A) = det(I) = 1$, therefore (because of Theorem **??**) $det(A^{-1})det(A) = 1$, since A is invertible $det(A) \neq 0$, and we find

$$det(A^{-1}) = \frac{1}{det(A)}$$

1.3 A Combinatorial Approach to Determinants

Observe that by expansion along the first row, we get

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{31}a_{23} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22}$$

Find that in each term of the determinant there is exactly one entry from each row and each column. Also find that all possible terms are included with the calculation of the determinant.

This can be used to find the determinant differently than by expansion along a row or column. Some of the terms are added some are subtracted, in order to determine the signs we will discuss permutations and their inversions.

Definition 3
A permutation of the numbers $\{1, 2, \ldots, n\}$ is an arrangement of these integers in some order without omission or repetition.

Example 11
$(3, 1, 2)$ is a permutation of $\{1, 2, 3\}$, but
$(1, 3, 3)$ is not a permutation of $\{1, 2, 3\}$ because 2 is missing and 3 is repeated.

There are $n!$ different permutations of $\{1, 2, \ldots, n\}$.

Definition 4
Let $(j_1, j_2, \ldots, j_n\}$ denote a permutation of $\{1, 2, \ldots, n\}$.
An inversion is said to occur in $(j_1, j_2, \ldots, j_n)$, whenever a larger integer precedes a smaller one.

The number of inversion in the elementary products will determine the sign used in the determinant. The total number of inversions in $(j_1, j_2, \ldots, j_n)$ is best found by

(1) Find the number of integers that are smaller than j_1 that follow j_1

(2) find the number of integers that are smaller than j_2 and follow j_2

Continue the counting process for the remaining entries $j_3, \ldots, j_{n-1}$.

The sum of these numbers is the total number of inversions in the permutation.

Example 12
$(5, 2, 6, 3, 1, 4$ has 4+1+3+1+0=9 inversions.
$(1, 2, 3)$ has 0 inversions.

Definition 5
A permutation is called odd(even), if the total number of inversions is an odd(even) integer.

Definition 6
Let A be a square matrix, then an elementary product from A is any product of n entries from A, where no two come from the same row or column or

$$a_{1j_1}a_{j_2}\cdots a_{nj_n}$$

where $(j_1, j_2, \ldots, j_n)$ is a permutation of $\{1, 2, \ldots, n\}$.
A signed elementary product is

$$\begin{array}{c} a_{1j_1}a_{j_2}\cdots a_{nj_n} \text{ if } (j_1, j_2, \ldots, j_n) \text{ is even} \\ -a_{1j_1}a_{j_2}\cdots a_{nj_n} \text{ if } (j_1, j_2, \ldots, j_n) \text{ is odd} \end{array}$$

Theorem 15
Let A be a square matrix, then $det(A)$ is the sum of all signed elementary products

Example 13

1. Let

$$A = \begin{bmatrix} 5 & 2 \\ 1 & 3 \end{bmatrix}$$

then

$$det(A) = 5(3) - 1(2) = 13$$

the number of inversions for the first term is 0 so, elementary product is even, and the number of inversions for the second term is 1 the elementary product is odd.

2. Let

$$A = \begin{bmatrix} 5 & 2 & -1 \\ 1 & 3 & 4 \\ -3 & -2 & 0 \end{bmatrix}$$

Then the elementary products are

products	permutation	inversions	odd/even
5(3)(0)	(1,2,3)	0	even
5(4)(-2)	(1,3,2)	1	odd
2(1)(0)	(2,1,3)	1	odd
2(4)(-3)	(2,3,1)	2	even
(-1)(1)(-2)	(3,1,2)	2	even
(-1)(3)(-3)	(3,2,1)	3	odd

$$det(A) = 0 - (-40) - 0 + (-24) + 2 - 9 = 9$$

or develop along the last row, then

$$det(A) = (-1)(-2 + 9) - 4(-10 + 6) = 9$$