0.1 Cross Product

The dot product of two vectors is a scalar, a number in $\mathbb{R}$.
Next we will define the cross product of two vectors in 3-space. This time the outcome will be a vector in 3-space.

Definition 1
If $\mathbf{u} = (u_1, u_2, u_3)$, and $\mathbf{v} = (v_1, v_2, v_3)$ are vectors in 3-space, then the cross product, $\mathbf{u} \times \mathbf{v}$, is the vector defined by

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - v_2u_3, u_3v_1 - v_3u_1, u_1v_2 - u_2v_1)$$

or in determinant notation

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Example 1
Let $\mathbf{u} = (1, -1, 2)$, and $\mathbf{v} = (-2, 3, 4)$, then

$$\mathbf{u} \times \mathbf{v} = ((-1)4 - (3)2, -[1(4) - (-2)2], 1(3) - (-2)(-1)) = (-10, -8, 1)$$

Theorem 1
If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space, then

(a) $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$, i.e. $\mathbf{u}$ is orthogonal to $\mathbf{u} \times \mathbf{v}$

(b) $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$

(c) $||\mathbf{u} \times \mathbf{v}|| = ||\mathbf{u}||^2 \; ||\mathbf{v}||^2 - (\mathbf{u} \cdot \mathbf{v})^2$ (Lagrange's Identity)

(d) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ (Relationship between dot and cross product)

(e) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$

Proof:

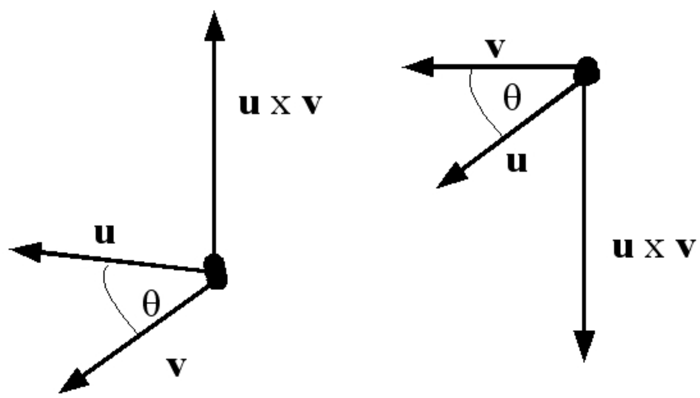
(a) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\begin{aligned} \mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) &= (u_1, u_2, u_3) \cdot (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1) \\ &= u_1(u_2v_3 - u_3v_2) - u_2(u_1v_3 - u_3v_1) + u_3(u_1v_2 - u_2v_1) \\ &= 0 \end{aligned}$$

Proof part (c) for bonus marks.

Right-hand rule
Parts (a) and (b) indicate that $\mathbf{u} \times \mathbf{v}$ is a vector orthogonal to $\mathbf{u}$ and $\mathbf{v}$.
Imagine the plane determined by the vector $\mathbf{u}$ and $\mathbf{v}$, then $\mathbf{u} \times \mathbf{v}$ can have potentially two orientations (up or down).
The right-hand rule is a tool for determining the orientation of $\mathbf{u} \times \mathbf{v}$.
Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, and suppose $\mathbf{u}$ is rotated through the angle θ until is overlaps $\mathbf{v}$. If the finger of the right hand are pointing in the direction of the rotation of $\mathbf{u}$ then the thumb indicates the orientation of $\mathbf{u} \times \mathbf{v}$.

Right-hand Rule



Theorem 2
If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space and $k \in \mathbb{R}$, then

(a) $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$

(d) $k(\mathbf{u} \times \mathbf{v}) = (k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v})$

(e) $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$

(f) $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

Proof: Exercise

Example 2
Standard unit vectors
The vectors

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$$

are called standard unit vectors in 3-space. They have norm one and lie along the coordinate axes. Every vector $\mathbf{u} = (u_1, u_2, u_3)$ can be expressed in terms of the standard unit vectors.

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$$

For example

$$(2, -3, 1) = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$$

The cross products of the standard unit vectors follow a given pattern

$$\mathbf{i} \times \mathbf{j} = \left(\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}, -\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \right) = (0, 0, 1) = \mathbf{k}$$

Similar we find

$$\begin{array}{lll} \mathbf{i}\times\mathbf{i}=\mathbf{0} & \mathbf{i}\times\mathbf{j}=\mathbf{k} & \mathbf{i}\times\mathbf{k}=-\mathbf{j} \\ \mathbf{j}\times\mathbf{i}=-\mathbf{k} & \mathbf{j}\times\mathbf{j}=\mathbf{0} & \mathbf{j}\times\mathbf{k}=\mathbf{i} \\ \mathbf{k}\times\mathbf{i}=\mathbf{j} & \mathbf{k}\times\mathbf{j}=-\mathbf{i} & \mathbf{k}\times\mathbf{k}=\mathbf{0} \end{array}$$

Observe the cross product of two consecutive vectors equals the following vector using the order

$$\mathbf{i}\rightarrow\mathbf{j}\rightarrow\mathbf{k}\rightarrow\mathbf{i}$$

When going in reverse direction the cross product of the two vector is the negative of the preceding vector.

Symbolically the cross product can be represented in determinant form

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

For example, for $\mathbf{u}=(1,2,0)$ and $\mathbf{v}=(-2,3,-1)$

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ -2 & 3 & -1 \end{vmatrix} = (-2)\mathbf{i}-(-1)\mathbf{j}+7\mathbf{k}=(-2,1,7)$$

In general

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w})\neq(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$$

Geometric interpretation of $||\mathbf{u}\times\mathbf{v}||$

The norm of the cross product of two vectors in 3-space gives the area of the parallelogram determined by the vectors.

Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in 3-space, then according to Lagrange's identity

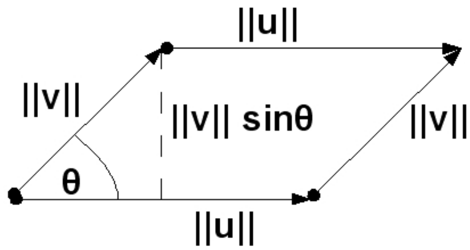
$$||\mathbf{u}\times\mathbf{v}||^2 = ||\mathbf{u}||^2\;||\mathbf{v}||^2-(\mathbf{u}\cdot\mathbf{v})^2$$

If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then according to the definition of the dot product $\mathbf{u}\cdot\mathbf{v} = ||\mathbf{u}||\;||\mathbf{v}||\cos(\theta)$ and therefore

$$\begin{aligned} ||\mathbf{u}\times\mathbf{v}||^2 &= ||\mathbf{u}||^2\;||\mathbf{v}||^2-||\mathbf{u}||^2\;||\mathbf{v}||^2\cos^2(\theta) \\ &= ||\mathbf{u}||^2\;||\mathbf{v}||^2(1-\cos^2(\theta)) \\ &= ||\mathbf{u}||^2\;||\mathbf{v}||^2\sin^2(\theta) \end{aligned}$$

Since the angle between two vectors is defined to fall between 0 and $\pi(180^o)$, $\sin(\theta)\geqslant 0$, therefore

$$||\mathbf{u}\times\mathbf{v}|| \;=\; ||\mathbf{u}||\;||\mathbf{v}||\sin(\theta)$$



Since $||\mathbf{v}||\sin(\theta)$ is the altitude of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$, thus the area A of the parallelogram is

$$A = (base)(altitude) = ||\mathbf{u}||\ ||\mathbf{v}||\sin(\theta) = ||\mathbf{u} \times \mathbf{v}||$$

Example 3
The area A of the triangle determined by points $P_1(1,-1,0)$, $P_2(2,4,0)$, and $P_3(0,-2,4)$ is equal to half the area of the parallelogram determined by vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$, therefore

$$\begin{aligned} A &= \frac{1}{2}||\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}|| \\ &= \frac{1}{2}||(2-1, 4-(-1), 0-0) \times (0-1, -2-(-1), 4-0)|| \\ &= \frac{1}{2}||(1,5,0) \times (-1,-1,4)|| \\ &= \frac{1}{2}||(20,-4,4)|| \\ &= \frac{\sqrt{20^2+(-4)^2+4^2}}{2} = \frac{\sqrt{432}}{2} \end{aligned}$$

Definition 2
If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space then the scalar triple product is defined as

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

Theorem 3
If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space then

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof:
For $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$

$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) &= \mathbf{u}\cdot\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\mathbf{k}\right) \\
&= u_1\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \\
&= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}
\end{aligned}$$

From the theorem follows: If $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors in 3-space then

$$\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = \mathbf{w}\cdot(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})$$

(interchanging two rows changes the sign of a determinant, thus interchanging two rows twice leads to the same determinant.

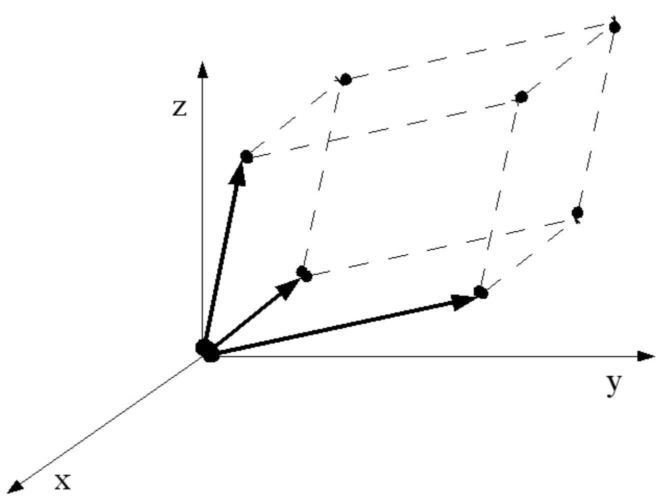
Theorem 4

(a) If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are vectors in 2-space, then the area A of the parallelogram determined by the vectors:

$$A = \left| det\begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}\right|$$

(b) If $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in 3-space then the volume V of the parallelepiped is

$$V = \left| det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}\right| = |\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$$



Theorem 5
If the vectors $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ are vectors in 3-space with the same initial point, then they lie in the same plane if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 0$$

This is a conclusion from the absolute value of the triple scalar product being the volume of the parallelepiped. The volume is zero if and only if the vectors lie in the same plane.

Remark:
The cross product is independent from the coordinate system.
The cross product is defines based on the coordinates of the vectors, but vectors were introduced independent from their coordinates raising the question if the cross product depends on the coordinates system applied.
Fortunately the answer is that the cross product is independent from the coordinate system.
The cross product of vectors **u** and **v** the vector

- which is orthogonal to **u** and **v**
- its orientation is determined by the right hand rule,
- and length $||\mathbf{u}||\ ||\mathbf{v}|| \sin(\theta)$.