MATH – 125 Practice Final

You have 2 hours to solve 6 problems of total value 35 points. Show your work; explain your solutions. Calculators, books, notes, and formulae sheets are not allowed on the exam.

1. Four vectors from R^4 are given: $\boldsymbol{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, and $\boldsymbol{d} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

   a). Determine if these vectors are a basis in R^4 (**Substantiate your answer!**).

   b). If they are, use Cramer's rule to find the components of $\boldsymbol{v} = \begin{bmatrix} 0 \\ 0 \\ 5 \\ 0 \end{bmatrix}$ in that basis.

   If these vectors are not a basis, state the dimension of the subspace spanned by them.

2. For each of the following sets, determine whether it is a subspace of the corresponding space. Substantiate your answers.

   a) The set S of all invertible matrices in M^{22} (the space of all 2×2 matrices with ordinary addition and multiplication by numbers).

   b) The set S of all polynomials $\boldsymbol{p}(x) = ax^2 + bx$ in P_2 (the space of all polynomials of degree less or equal 2).

3. T is a transformation defined as follows: $T(\boldsymbol{r}) = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ y-x \end{bmatrix}$.

   a) What is the domain and the codomain of T?
   b) Is T linear?
   c) Find the standard matrix of T if possible.
   d) Is T a one-to-one transformation? Is T a transformation onto? – Explain.

4. For each of the following transformations T find the standard matrix and determine whether it is invertible.

a) T is the projection of R^2 onto the line $y = -x$.

b) T is the reflection of R^3 in the plane $x - 2y + z = 0$.

5. Four vectors in R^4 are given:

$$\boldsymbol{a}=\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \boldsymbol{b}=\begin{bmatrix}0\\1\\-1\\3\end{bmatrix}, \boldsymbol{c}=\begin{bmatrix}-1\\1\\-2\\3\end{bmatrix}, \text{ and } \boldsymbol{d}=\begin{bmatrix}1\\1\\0\\4\end{bmatrix}$$

Find a basis in the subspace spanned by these vectors and state the dimension of this subspace.

6. For each of the following statements, determine whether it is true (T) or false (F). Substantiate your decision (prove or disprove the statement).

a) If A, B and C are such matrices that $AC = AB$, then $C = B$.

b) The following equality holds for any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$: $\|\boldsymbol{u}+\boldsymbol{v}\|^2 = \|\boldsymbol{u}\|^2 + \|\boldsymbol{v}\|^2$.

c) The determinant of a matrix will not change if its first row is being replaced with a linear combination of the first, second, and third rows of this matrix.

d) The line L given by the equations $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+3}{2}$ is parallel to the plane P defined by its normal equation $-4x + 4y - 8z + 5 = 0$.

e) If A is such a 5×5 matrix that $\det A \neq 0$, then the system $A\boldsymbol{x} = \boldsymbol{0}$ has a nontrivial solution.