Solutions for practise final

1. Let V be a vector space, then $S \subset V$ is a subspace if
   (i) for $\vec{x}, \vec{y} \in S$: $\vec{x} + \vec{y} \in S$
   (ii) for $\vec{x} \in S$, $k \in \mathbb{R}$: $k\vec{x} \in S$

2. Let $A \in \mathbb{R}^{n \times n}$ for some n, then $A^{-1} \in \mathbb{R}^{n \times n}$ is the inverse of A if

$$A A^{-1} = A^{-1} A = I_n$$

3. No. $\vec{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in the set ($1^2 + 0^2 \leq 1$), but

$5i = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$ is not because

$$5^2 + 0^2 = 25 \not\leq 1$$

4. $\vec{u}, \vec{v}$ orthogonal $\Rightarrow \vec{u} \cdot \vec{v} = \vec{0} = \vec{v} \cdot \vec{u}$

$$\|\vec{u} + \vec{v}\|^2 = (\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})$$

$$= \vec{u} \cdot \vec{u} + \underbrace{\vec{u} \cdot \vec{v}}_{=0} + \underbrace{\vec{v} \cdot \vec{u}}_{=0} + \vec{v} \cdot \vec{v}$$

$$= \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} = \|\vec{u}\|^2 + \|V\|^2$$

5. Let V be a vector space, $\vec{v} \in V$

$$0\vec{v} = (1+(-1))\vec{v} = 1\cdot\vec{v} + (-1)\cdot\vec{v}$$

$$= \vec{v} + (-\vec{v}) = \vec{0}$$

property of vector space $1\cdot\vec{v} = \vec{v}$ + theorem $(-1)\vec{v} = -\vec{v}$

property of vector space

6. (a) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 0 & 4 & 2 \\ 0 & -1 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -4 & -2 & 0 \\ 0 & -1 & 1 & 4 \end{bmatrix} \xrightarrow{R_2 \updownarrow R_3}$

$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 1 & 4 \\ 0 & -4 & -2 & 0 \end{bmatrix} \xrightarrow{R_3 - 4R_2} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -6 & -16 \end{bmatrix}$

(b) The REF of A has 3 pivots

$\Rightarrow \operatorname{rank}(A) = 3$

(c) Yes, $\operatorname{rank}(A) = 3 \Rightarrow$ span (columns of A) $= \mathbb{R}^3 \Rightarrow A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \in \mathbb{R}^3$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -6 \\ -16 \end{bmatrix} \right\}$

(f)

$x_4 = t \,, \; t \in \mathbb{R}$

$x_3 = \frac{16}{6} t \,, \quad x_2 = \frac{16}{6} t + 4t = \frac{40}{6} t$

$x_1 = -\frac{80}{6} t - \frac{48}{6} t - \frac{6}{6} t = -\frac{134}{6} t$

Basis of Null Space of A:

$$\begin{bmatrix} -134/6 \\ 40/6 \\ 16/6 \\ 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} -134 \\ 40 \\ 16 \\ 1 \end{bmatrix}$$

7.

(a) $$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3/-1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\xrightarrow[R_1 - 3R_3]{R_2 - 4R_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \xrightarrow{R_1 - 2R_2}$$

$$\left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} & \underbrace{\begin{matrix} 1 & -2 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{matrix}}_{A^{-1}} \end{array}\right], \quad A^{-1} = \begin{bmatrix} 1 & -2 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

b) $\det(A) = 1 \cdot 1 \cdot (-1) = -1$

c)

$$x_3 = \frac{\det A_3}{\det A}$$

$$\det A_3 = \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = 3$$

$$\Rightarrow \quad x_3 = \frac{3}{-1} = -3$$

8)

a) $$\vec{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad s, t \in \mathbb{R}$$

b) $\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cos(\theta)$

$$\vec{u} \cdot \vec{v} = 0 + 2 - 3 = -1$$

$$\|\vec{u}\| = \sqrt{16 + 4 + 9} = \sqrt{29}$$

$$\|\vec{v}\| = \sqrt{0 + 1 + 1} = \sqrt{2}$$

$$\Rightarrow \quad \cos(\theta) = \frac{-1}{\sqrt{29}\,\sqrt{2}} = \frac{-1}{\sqrt{58}}$$

c)

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}$$

$$= \frac{-1}{2} \cdot \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

d)

$$L: \vec{x} = \vec{p}_0 + t \cdot \vec{d} \quad , t \in \mathbb{R}$$

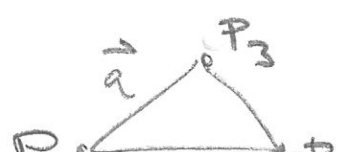
$$\vec{d} = \vec{u} \times \vec{v} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -4 & 2 & 3 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= -5\vec{i} - 4\vec{j} + (-4)\vec{k} = \begin{bmatrix} -5 \\ -4 \\ -4 \end{bmatrix}$$

$$L: \vec{X} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} + t \begin{bmatrix} -5 \\ -4 \\ -4 \end{bmatrix}, \quad t \in \mathbb{R}$$

9.

a)



$$\Rightarrow A = \frac{1}{2} \|(\vec{a} \times \vec{b})\| \qquad \vec{a} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$$

$$= \left| \frac{1}{2} \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & -4 \\ -3 & -1 & -2 \end{matrix} \right\| \right| \qquad \vec{b} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2} \| 2\vec{i} - 12\vec{j} + 3\vec{k} \|$$

$$= \frac{1}{2} \sqrt{4 + 144 + 9} = \frac{1}{2}\sqrt{157}$$

(5) $\vec{N} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ $\quad \vec{p}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

$$\Rightarrow d = \text{proj}_{\vec{N}} \vec{p}_1 = \frac{\vec{N} \cdot \vec{p}_1}{\| \vec{N} \|}$$

$$= \frac{(-1 - 2 + 6)}{\sqrt{1 + 1 + 9}} = \frac{3}{\sqrt{11}}$$

10. $(\vec{u} + k\vec{v}) \times (\vec{u} - l\vec{v})$

$$= \underbrace{\vec{u} \times \vec{u}}_{=0} - l\vec{u} \times \vec{v} + k \underbrace{\vec{v} \times \vec{u}}_{-\vec{u} \times \vec{v}} - lk \underbrace{\vec{v} \times \vec{v}}_{=0}$$

$$= - l\, \vec{u} \times \vec{v} - k\, \vec{u} \times \vec{v}$$

$$= -(l + k)(\vec{u} \times \vec{v})$$

11.

a) $T(\vec{e}_1) = \vec{e}_1 \quad T(\vec{e}_2) = \vec{e}_2$ 7

$T(\vec{e}_3) = -\vec{e}_3$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

b)

$$[R] = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

c)

$$[T \circ R] = [T][R] = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

d) $$[T \circ R] \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3/2 \end{bmatrix}$$

12)

$$\begin{bmatrix} -4 & 0 & -4 \\ 2 & 1 & 0 \\ 3 & -1 & 5 \end{bmatrix} \xrightarrow{R_1/-4} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & 5 \end{bmatrix}$$

$$\xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 + R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$B = \left\{ \begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis.

b) Should ask, if the basis of (a) is a basis of $\mathbb{R}^3$.

No, $\dim \mathbb{R}^3 = 3 \rightarrow$ a basis has to have 3 vectors.

13) a) No. Example $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\rightarrow$ exactly one solution $\vec{x} = \vec{0}$

b) A matrix, B, is symmetric if $B^T = B$

$(AA^T)^T = (A^T)^T A^T = AA^T$

$\Rightarrow AA^T$ is symmetric

c) No,

$(AB)^2 = ABAB$