MATH-120 Midterm 1 (Fall 2007) Time: 50 min Instructor: M.Solomonovich

Section	Last name	Given name(s)	ID #
	SOLUTIONS		

You have 50 min. to solve 5 problems; the value of each problem is 5 points. Show your work; substantiate your solutions where necessary. Calculators, notes, formula sheets are not allowed on this exam.

1. In each of the following cases evaluate the expression or briefly explain why it is not defined.

a) $\begin{bmatrix} 1 & 8 & 0 \\ -2 & 5 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ Not defined:

2×3 2×2 2×1 The addition is defined only for the matrices of the same size

b) $([-1 \ \ 2 \ \ 5]+3[1 \ \ 0 \ \ 2])\cdot\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = [2 \ \ 2 \ \ 11]\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = [2-2+22] = [22]$

c) $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}\cdot[0 \ \ 1 \ \ 4] = \begin{bmatrix} 0 & 2 & 8 \\ 0 & -1 & -4 \\ 0 & 3 & 12 \end{bmatrix}$.

2. Find all the values of the parameters a and b such that the following linear system (i) is inconsistent; (ii) has infinitely many solutions; (iii) has exactly one solution.

$$\begin{aligned} 2x-7y+3z&=b \\ x-5y+4z&=-3 \\ -2x+y+az&=0 \end{aligned} \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 2 & -7 & 3 & b \\ -2 & 1 & a & 0 \end{array}\right] \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & 3 & -5 & b+6 \\ 0 & -9 & a+8 & -6 \end{array}\right] \begin{array}{l} R_2 \to \frac{1}{3}R_2 \\ R_3 \to R_3 + 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 4 & -3 \\ 0 & 1 & -\frac{5}{3} & \frac{b}{3}+2 \\ 0 & 0 & a-7 & 3b+12 \end{array}\right] \Longrightarrow (a-7)z = 3(b+4)$$

(i) if $\boxed{a=7 \text{ and } b\neq -4}$, the last equation is $0 \cdot z = \text{nonzero}$, hence the system has no solutions.

(ii) if $\boxed{a=7 \text{ and } b=-4}$, the last equation $0 \cdot z = 0$ is satisfied for any value of z, therefore there are ∞ many solutions.

(iii) if $\boxed{a\neq 7}$, $z=\dfrac{3b+12}{a-7}$ and by back substitution the unique solution can be found.

3.

a) Evaluate the following determinant:

$$\begin{vmatrix} -2 & 0 & 3 & 1 \\ -1 & 1 & 0 & 4 \\ 0 & 2 & -1 & 0 \\ 1 & 2 & 1 & 4 \end{vmatrix} = \left(\begin{array}{l}\text{expansion} \\ \text{along the } 3^{\text{rd}} \text{ row}\end{array}\right) =$$

$$= 2\cdot(-1)^{3+2}\begin{vmatrix} -2 & \boxed{3} & 1 \\ -1 & \boxed{0} & 4 \\ 1 & \boxed{1} & 4 \end{vmatrix} + (-1)(-1)^{3+3}\begin{vmatrix} \boxed{-2 \quad 0 \quad 1} \\ -1 \quad 1 \quad 4 \\ 1 \quad 2 \quad 4 \end{vmatrix} =$$

$$= (-2)\left[3(-1)\begin{vmatrix} -1 & 4 \\ 1 & 4 \end{vmatrix} + 1\cdot(-1)\begin{vmatrix} -2 & 1 \\ -1 & 4 \end{vmatrix}\right] - \left[(-2)\begin{vmatrix} 1 & 4 \\ 2 & 4 \end{vmatrix} + 1\cdot\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}\right] =$$

$$= -2\left((-3)\cdot(-8) - (-8+1)\right) - \left((-2)\cdot(-4) - 3\right) =$$

$$= -2(24+7) - 5 = -67.$$

b) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -8 \\ 2 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & 2 \\ 1 & -2 \end{bmatrix}$.

One can check (you don't have to) that $AB = AC$ although $B \neq C$.
Explain, why this is possible.

$AB = AC$ implies $B = C$ only if A^{-1} exists: $\underbrace{A^{-1}\cdot A}_{I}B = \underbrace{A^{-1}\cdot A}_{I}C \Rightarrow B = C$.

Otherwise, it may happen that $B \neq C$.

In this case A^{-1} does not exist:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3R_1]{} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

4. Find all the matrices that commute with $A = \begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix}$

(i) under addition
(ii) under multiplication.

(i) Any 2×2 matrix commutes with A under addition

(ii) Let X be such a matrix that

$$\underset{2\times 2}{A}\,\underset{2\times n}{X} = \underset{m\times 2}{X}\,\underset{2\times 2}{A}.$$

AX is defined if X has 2 rows.
XA is defined if X has 2 columns.
Then, X must be 2×2.

Let $X = \begin{bmatrix} x & y \\ z & t \end{bmatrix}$

$$AX = \begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & t \end{bmatrix} = \begin{bmatrix} -x+z & -y+t \\ 2x+4z & 2y+4t \end{bmatrix}$$

$$XA = \begin{bmatrix} x & y \\ z & t \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -x+2y & x+4y \\ -z+2t & z+4t \end{bmatrix}$$

$$\begin{cases} -x+z = -x+2y \\ -y+t = x+4y \\ 2x+4z = -z+2t \\ 2y+4t = z+4t \end{cases} \Rightarrow \begin{aligned} 2y - z &= 0 \\ x+5y \quad -t &= 0 \\ 2x + \quad 5z-2t &= 0 \\ 2y - z &= 0 \end{aligned} \quad R_1 \leftrightarrow R_2, \quad R_3 \to R_3 - 2R_2, \quad R_4 \to R_4 - R_1$$

$$\to \left.\begin{aligned} x+3y \quad -t &= 0 \\ -10y+5z &= 0 \\ 2y - z &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x+3y-t = 0 &\Rightarrow x = -3y+t \\ y - \tfrac{1}{2}z = 0 &\Rightarrow y = \tfrac{1}{2}z \end{aligned}$$

$$\begin{cases} x = -\frac{3}{2}z + t \\ y = \frac{1}{2}z \end{cases}$$

Any matrix of the type

$$X = \begin{bmatrix} -\frac{3}{2}z+t & \frac{1}{2}z \\ z & t \end{bmatrix}$$

will commute with A under multiplication.

5. For each of the following statements decide whether it is true (T) or false (F). Substantiate your decision.

a) A system is consistent if the number of equations does not exceed the number of unknowns (variables) of the system.

FALSE: $\left.\begin{aligned} x - 2y + z &= 1 \\ 2x - 4y + 2z &= 5 \end{aligned}\right\}$ is inconsistent.

b) If the coefficient matrix of a homogeneous system with 5 equations for 5 unknowns is invertible, the system will have only a trivial solution.

True: $AX = 0 \Rightarrow A^{-1}AX = A^{-1}0 \Rightarrow$

$\Rightarrow X = 0.$

c) If A and B are such matrices that $AB = O$ then $A = O$ or $B = O$ or $A = B = O$.

FALSE. For example, the system

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\left(A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, B = \begin{bmatrix} x \\ y \end{bmatrix} \right)$

will be satisfied for any x & y such that $x + 2y = 0$. For example,

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -6 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$